16th World Congress in Computational Mechanics (WCCM)
21-26 July 2024, Vancouver, Canada
A. Korobenko, M. Laforest, S. Prudhomme, R. Vaziri (Eds)

APPLICATION OF THE HARMONIC BALANCE METHOD TO PREDICT WAVE PROPAGATION IN ONE-DIMENSIONAL NONLINEAR METAMATERIAL EXCITED HARMONICALLY

RANGEL M. BARBOSA[1] AND ALBERTO L. SERPA[2]

[1] Department of Computational Mechanics, School of Mechanical Engineering, UNICAMP
Mendeleyev Street, 200 - Campinas - SP
rangelmb@yahoo.com.br

[2] Department of Computational Mechanics, School of Mechanical Engineering, UNICAMP
Mendeleyev Street, 200 - Campinas - SP
alserpa@unicamp.br

Key words: Harmonic Balance, Nonlinear Dynamics, Periodic Materials.

Abstract. This work exposes a computational procedure designed to aid in modeling mechanical systems featuring stiffness nonlinearity. The basis of the procedure is the Harmonic Balance Method, which is combined with a numerical continuation technique. To present the efficacy of the approach, a one-dimensional nonlinear metamaterial is analyzed. The aim is to demonstrate the suitability of the procedure to extract information regarding higher harmonic generation and the influence of the amplitude of excitation on the system dynamic response.

1 INTRODUCTION

Periodicity and local resonance are features managed to achieve efficient metamaterials. In the vast majority, the latter aspect is studied by applying linear resonators. Recently, due to dynamic characteristics provided by nonlinearity, such as the dependence of the response on the excitation amplitude and wave energy transfer to higher harmonics, nonlinear resonators have been considered [1]. However, nonlinear behavior is typically a challenging aspect to address. It often increases complexity and must be carefully analyzed at the design stage to avoid trial-and-error approaches in the design of mechanical devices [2].

This requirement is pursued in this work, where the objective is to develop a computational procedure capable of capturing some nonlinear behaviors in a one-dimensional metamaterial chain, such as the emergence of higher harmonics in the response and the influence of excitation amplitude. The Harmonic Balance Method (HBM) is the basis of the procedure mentioned above, which also includes a numerical continuation technique considering the pseudo-arc-length and Newton method.

Initially, the implementation of the HBM method is validated on a 2-DOF system with nonlinear cubic stiffness. Subsequently, a system with 10 unit cells is analyzed, each cell composed of a main chain mass connected to a local oscillator by a nonlinear cubic spring, a linear spring, and a viscous damper. Between consecutive unit cells, there is a linear spring and a viscous damper. Considering the periodicity provided by the composition of unit cells, such a system may be defined as a metamaterial [1, 3].

Applying a harmonic force to the first unit cell aims to observe the vibration transmissibility throughout the structure and changes in behavior depending on force amplitude. Besides, observing the harmonics that compose the response should allow one to evaluate the energy transfer to higher harmonics.

2 NUMERICAL PROCEDURE

2.1 Harmonic balance for periodic solutions

Consider the system of differential-algebraic equations:

$$r(x, \dot{x}, \ddot{x}, t) = 0, \tag{1}$$

where r represents a function defined as

$$r(x, \dot{x}, \ddot{x}, t) = f_{lin}(x, \dot{x}, \ddot{x}) + f_{nl}(x, \dot{x}, t) - f_{ex}(t), \tag{2}$$

where x, $\dot{x}$, $\ddot{x}$ represent displacement, velocity and acceleration, respectively. f_{lin} represents the linear part, f_{nl} the nonlinear part, f_{ex} a periodic excitation imposed to the system, and t the time.

A periodic solution for this system can be expressed as

$$x(t) = x(t + T), \tag{3}$$

with $T > 0$.

In many cases, r represents a complicated nonlinear function that prevents the obtention of the exact solution. In such a situation, approximation methods are required, in which the solution can take the form of a linear combination as

$$x_h(t, \beta_k) = \sum_{k=0}^{H} \beta_k b_k(t), \tag{4}$$

where x_h is an approximated solution, and the index h indicates terms related to this approximation of the result of Eq.(2). b_k is ansatz or base functions, and β_k the coefficients.

As x_h is an approximation, naturally it does not satisfy Eq.(2) to all $t \in [0, T]$. Defining a set of ansatz functions and assuming Eq.(4) as solution produces a residual $r_h(t, \beta_k) \neq 0$, which depends on time and the coefficients. Hence, to get as close as possible to the exact solution, the goal of this approach becomes to determine the coefficients β_k. This procedure follows the weighted residual approach, which is used by most methods to approximate periodic solutions. This approach requires satisfying Eq.(2) in a weighted average sense, i.e.,

$$\frac{1}{T}\int_0^T \rho_j(t) r_h(t,\beta_k) dt = 0, \qquad j = 1, ..., H, \tag{5}$$

where ρ_j are the weight functions and H is the truncation order in Eq.(4).

In the HBM, Fourier base functions are used as ansatz as well as weight functions, which classify the weight residual approach as the Galerkin method. Because of this, HBM is sometimes called the Fourier-Galerkin method [2]. Defining $b_1 = 1, b_2 = cos(\omega t), b_3 = sin(\omega t)$ and so on, Eq.(4) can be rewritten as

$$x_h(t,\beta_k) = \beta_1 + \sum_{k=1}^{H} \beta_{2k} cos(k\omega t) + \beta_{2k+1} sin(k\omega t), \tag{6}$$

where ω is the angular frequency.

It can be observed that the total number of ansatz functions is $2H+1$, and the coefficients β correspond to the Fourier coefficients of x_h. Also, as x_h and its time derivate $\dot{x}_h$ are T-periodic, so is the residual r_h. Then, following the Galerkin approach and using the Fourier coefficients definition, HBM requires that the residual's Fourier coefficients vanish up to the ansatz's truncation order [4].

2.2 Application to mechanical systems

Considering a mechanical system, Eq.(2) can be rewritten as

$$M\ddot{x} + C\dot{x} + Kx + f_{nl}(x,\dot{x},t) = f_{ex}(t), \tag{7}$$

where M, C, and K are mass, damping, and stiffness matrices, respectively. Then, it is possible to consider that the linear part of the equation is composed by

$$f_{lin}(x,\dot{x},\ddot{x}) = M\ddot{x} + C\dot{x} + Kx. \tag{8}$$

If the approximation of Eq.(6) is substituted into Eq.(7), a residual is obtained. As stated in the last section, HBM requires that the Fourier coefficients of this residual vanish up to the truncation order of the ansatz. As presented by Krack *et al.* [4], the equation featuring these coefficients contains terms relative to f_{nl}, f_{ext}, and f_{lin}, according to

$$\hat{f}_{lin,H}(\hat{x}_h,\omega) + \hat{f}_{nl,H}(\hat{x}_h,\omega) - \hat{f}_{ex,H}(\omega) = 0, \tag{9}$$

where the hat indicates the Fourier coefficients.

The term $\hat{f}_{ex,H}$ is usually known, and obtaining the linear part is a straightforward task. Then, the main challenge stays on the nonlinear coefficients. In this work, the Alternating Frequency-Time scheme (AFT) is used to obtain the Fourier coefficients of the nonlinear part. The method, proposed by Cameron *et al.* [5], seeks to use the system's time response as a means to get the nonlinear coefficients in the frequency domain. This procedure can be presented as

$$\hat{x}_h \xrightarrow{\text{FFT}^{-1}} x_h(t) \rightarrow f_{nl}(x, \dot{x}, t) \xrightarrow{\text{FFT}} \hat{f}_{nl,H}, \tag{10}$$

which indicates that through the inverse Fourier transform of $\hat{x}_h$, the time signal $x_h(t)$ is obtained. Using the mathematical formulation describing the nonlinearity of the system on $x_h(t)$, the nonlinear force in the time domain is achieved $f_{nl}(x, \dot{x}, t)$. Applying the Fourier transform on the latter, results in the Fourier coefficients of the nonlinear part $\hat{f}_{nl,H}$.

2.3 Numerical continuation

Implementing the HBM involves root-finding steps, which require standard methods for approximating solutions, such as Newton method. However, as such methods break down at turning points because the Jacobian matrix becomes singular, continuation methods need to be employed together.

To access the evolution of the response amplitude under variation of frequency, numerical continuation is used considering the predictor-corrector technique, as in the analyses of interest turning points may be present. This approach predicts a solution advancing a distance Δs (step length) from a given initial solution. In the implemented approach, the step is taken over the Fourier coefficients and the excitation frequency.

Generally, the predicted point is not located on the solution branch, so this point is iteratively corrected using iterative methods until a residual criterion is satisfied and the new solution point is adopted. The aim is to generate a sequence of suitably spaced solutions within the given frequency range and go around turning points. Besides overcoming turning points, continuation increases the numerical robustness and efficiency. This is especially important in ranges near resonances [4].

This procedure considers the frequency as a free parameter and additional unknown, which makes the HBM's system of equations underdetermined. This issue is solved by using an additional equation, which determines where on the solution path the next solution point ends up and, in this sense, parameterizes the solution path.

This additional equation, also known as parametrization constraint, is defined in the implementation as

$$N(u(s), \lambda(s), s) = (u(s) - u_0)^T u_0' + (\lambda(s) - \lambda_0)^T \lambda_0' - \Delta s = 0, \tag{11}$$

where the prime represents differentiation with respect to the arclength variable s and λ is the free parameter. The form of Eq.(11), known as pseudo-arc-length method, is selected to approximate the usual arclength definition. Its graphical interpretation can be seen in Figure 1. In this study, the excitation frequency is taken as λ and the Fourier coefficients as u.

3 APPLICATION OF THE PROPOSED APPROACH TO MECHANICAL SYSTEMS

3.1 Validation

To validate the implementation of the HBM, a system of 2 degrees of freedom with cubic nonlinear stiffness is analyzed. Assuming that the system is weakly damped with weak nonlin-

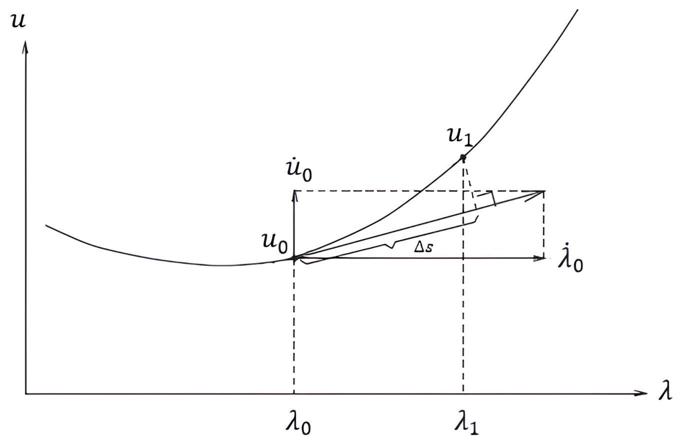

Figure 1: Graphical interpretation of pseudo-arc-length continuation.

ear stiffnesses and applying a low amplitude excitation force, the equation of motion takes the form

$$m\ddot{x}_1 + kx_1 + \epsilon c_1\dot{x}_1 + \epsilon k_{nl}x_1^3 + \epsilon k_{coup_nl}(x_1 - x_2)^3 = 2\epsilon P_1 cos(\omega t),$$
$$m\ddot{x}_2 + kx_2 + \epsilon c_2\dot{x}_2 + \epsilon k_{nl}x_2^3 + \epsilon k_{coup_nl}(x_2 - x_1)^3 = 2\epsilon P_2 cos(\omega t), \tag{12}$$

where $|\epsilon| \ll 1$. The parameters adopted for the analysis are $m = 1$, $k = 1$, $c_1 = 0.05$, $c_2 = 0.07$, $k_{nl} = 1$, $k_{coup_nl} = 0.1$, $P_1 = 0.2$, $P_2 = 0$ and $\epsilon = 0.01$.

Vakakis [6] has proposed and investigated this system using the method of multiple scales, and the results serve as a basis for comparison. The results obtained from HBM are presented in Figure 2, where the frequency detuning parameter is $\sigma = (\omega - 1)/\epsilon$, indicating the difference between the excitation frequency and the linearized natural frequencies of the system is used.

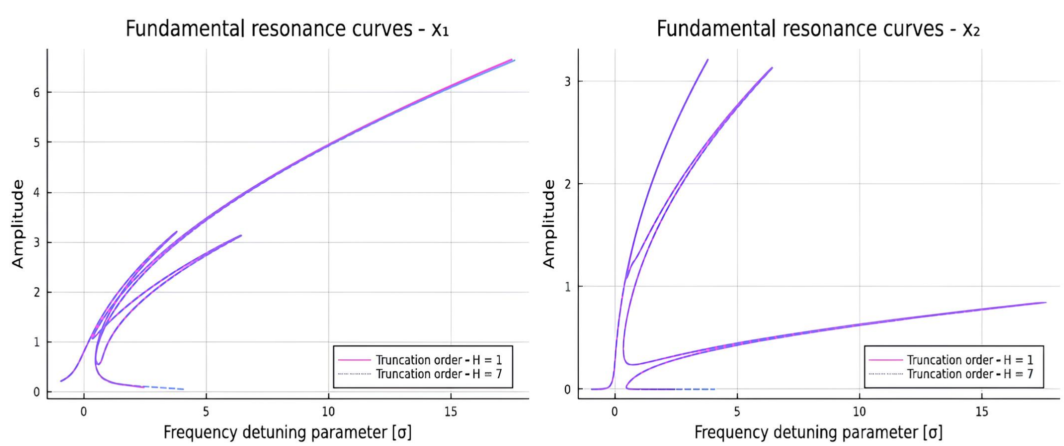

Figure 2: Fundamental resonance curves.

The harmonic truncation orders $H = 1$ and $H = 7$ were considered. Comparing these cases, slight variations are observed. It can be concluded that higher harmonics do not have considerable influences on the responses of this system and a lower truncation order can be adopted for the analysis.

The results show a similar shape to what is expected from [6]. Although not presented here, at values in which Vakakis[6] validated the results, HBM values are the same. Thus, it can be concluded that the implementation of the HBM worked properly.

3.2 Analysis of the chain of oscillators

The chain of oscillators analyzed is shown in Figure 3. It is composed of 10 unit cells, each containing a main mass (m) belonging to the chain and a second mass (m_0) representing the local resonator. The excitation force emerges as result of an imposed displacement to the spring at the left end of the system. It is represented by $F(t) = F_0 cos(\omega t)$, where $F_0 = (A_0 k)$, being the A_0 the amplitude of the imposed displacement and ω the angular frequency [3].

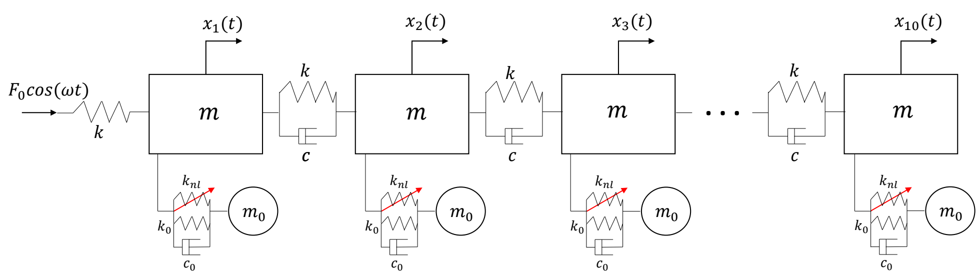

Figure 3: Oscillator chain model.

The equation of motion for the n^{th} unit cell can be given as

$$
\begin{aligned}
m\ddot{x}_n + k(2x_n - x_{n-1} - x_{n+1}) + c(2\dot{x}_n - \dot{x}_{n-1} - \dot{x}_{n+1}) + \\
k_0(x_n - y_n) + c_0(x_n - y_n) - k_{nl}(y_n - x_n)^3 = 0, \\
m_0\ddot{y}_n + k_0(y_n - x_n) + c_0(y_n - x_n) + k_{nl}(y_n - x_n)^3 = 0.
\end{aligned}
\tag{13}
$$

In the analyses $H = 5$ is used, and the parameters values are $m = 0.1Kg, c = 0.02Ns/m, k = 2400N/m, m_0 = 0.01Kg, c_0 = 0.08Ns/m, k_0 = 210N/m$ and $k_{nl} = 4000N/m^3$. Besides, the adopted amplitudes to the imposed displacement are $A_0 = 0.01m, A_0 = 0.008m$ and $A_0 = 0.005m$.

To graphically express the results, the frequency is taken as $\Omega = \frac{\omega}{\omega_0}$, where ω_0 is the resonator's natural frequency. Besides, the transmissibility curves are calculated as the ratio of the amplitude of displacement of the last main mass to the amplitude of the displacement imposed on the spring on the left end in Figure 3.

The first analysis using HBM on this system is presented in Figure 4, where $k_{nl} = 0$. In the absence of nonlinear stiffness, this system can be classified as linear. Therefore, its results are independent of the amplitude of the excitation force, and no higher harmonics are generated in its response. However, the resonances of the system and the effect of the resonator can be observed through this analysis. It can be observed that around $\Omega = 1$, the transmissibility value reduces considerably. This behavior is expected since this frequency corresponds to ω_0, the natural frequency of the resonator. As a result, the vibration of the mass in the main chain is considerably attenuated.

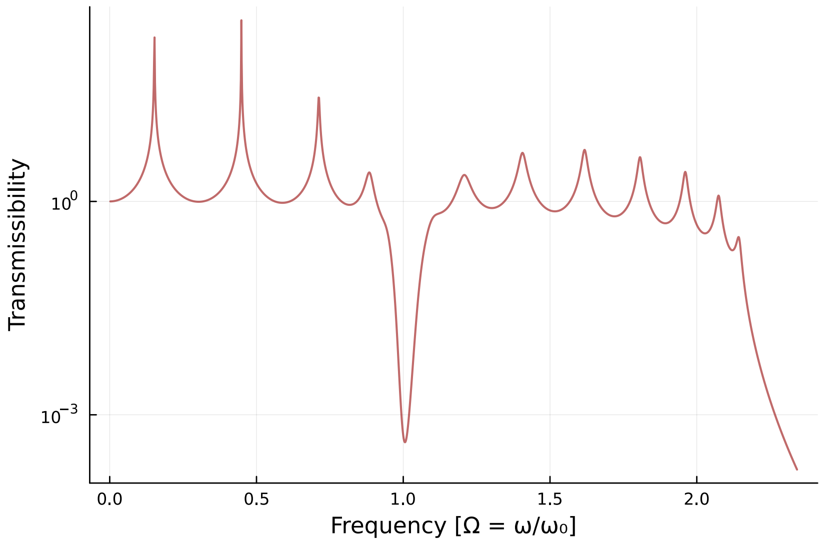

Figure 4: Transmissibility of the linear metamaterial.

As the nonlinear stiffness is present in the resonator, behavior changes concerning the nonlinearity tend to be more relevant near its natural frequency, $\Omega = 1$. Then, observing this and using the information provided from Figure 4, a frequency range is defined to observe the influence of the force amplitude and higher harmonic generation in the nonlinear system. The analysis is presented in Figure 5.

Comparing the curves, it can be observed that increasing the amplitude of the force has the effect to slightly shifts the band gap to the right and slightly decreases the level of displacement attenuation, as reported by Silva *et al.* [1]. Furthermore, the inclination of the peak of resonances is increased with the force amplitude. This behavior is expected from HBM analysis, as unstable responses are also extracted from the method.

Regarding higher harmonic generation, Figure 6 presents the ratio between first and third harmonics amplitudes, the latter is expected to occur due to cubic nonlinearity. It is possible to observe that higher force amplitudes lead to more relevant third harmonics amplitude values. From this, it is possible to infer that the differences between the responses in Figure 5 are induced by higher harmonic effects, which shows the relevance of such a mechanism in modeling nonlinear mechanical systems.

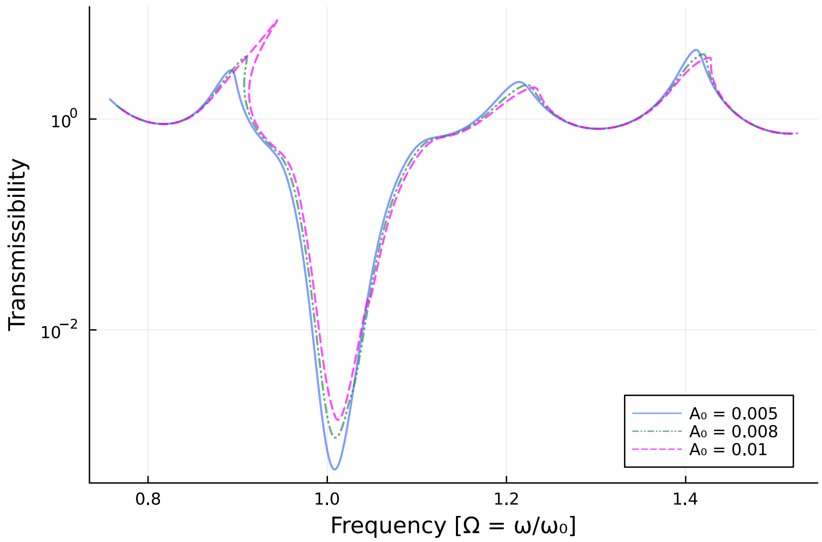

Figure 5: Transmissibility of the nonlinear metamaterial.

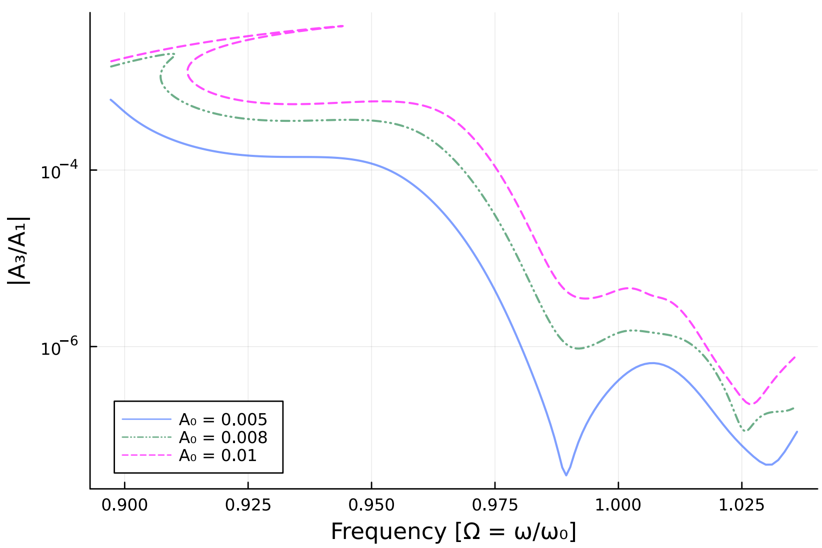

Figure 6: Third and first harmonics amplitude ratio.

4 CONCLUSIONS

This paper proposes analyzing a mechanical system using the harmonic balance method considering higher harmonics effects. First, a code was developed to simulate the system and tested on a two-degree-of-freedom mechanical system for which the response was known. This first step made it possible to validate the code. Subsequently, a metamaterial composed of 10 unit cells was analyzed. At this stage, the proposed procedure led to expected results, such as higher harmonic generation and the dependence of the system response to the amplitude of the force, indicating a trend to nonlinear behaviors even more prominent with higher excitation amplitudes. Therefore, the results indicate the suitability of the adopted approach for such analysis. It is expected that it may aid in further analyses of nonlinear mechanical systems.

Acknowledgements The authors thank Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP), Grant Number 2017/15736-3, Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq), Grant Number 307954/2022-6, and Coordenação de Aperfeiçoamento de Pessoal de Nível Superior (CAPES), Grant Number 88887.817298/2023-00, for the financial support.

REFERENCES

[1] Silva, P.B., Leamy, M.J., Geers, M.G.D. and Kouznetsoval, V.G. Emergent subharmonic band gaps in nonlinear locally resonant metamaterials induced by autoparametric resonance. *Phys. Rev. E*. (2019).

[2] Detroux, T., Renson, L., Masset, L. and Kerschen, G. The Harmonic Balance Method for Bifurcation Analysis of Large-Scale Nonlinear Mechanical Systems. *Comp. Method Appl. M*. (2015) p.18-38.

[3] Yu, M., Fang, X. and Yu, D. Combinational design of linear and nonlinear elastic metamaterials. *Int. J. Mech. Sci*. (2021).

[4] Krack, M. and Gross, J. *Harmonic balance for nonlinear vibration problems*. Springer, (2019).

[5] Cameron T. M. and Griffin J. H. An Alternating Frequency/Time Domain Method for Calculating the Steady-State Response of Nonlinear Dynamic Systems. *J. Appl. Mech*. (1989) p.149-154.

[6] Vakakis A. F., Fundamental and Subharmonic Resonances in a System with a '1-1' Internal Resonance. *Nonlinear Dyn*. (1992) p.123-143.

[7] Nayfeh A. H., Mook D. T., *Nonlinear Oscillations*. John Wiley and Sons, (1995).