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GEOMETRIC FORMULATION OF THREE-TEMPERATURE RADIATION HYDRODYNAMICS

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Abstract. Three-temperature (3T) radiation hydrodynamics models high energy-density plasma of nonlinearly coupled electron, ion, and radiation fields, finding applications in astrophysics and inertial confinement fusion. We present a geometric formulation of three-temperature radiation hydrodynamics. This is done utilizing an irreverisble port-Hamiltonian framework in the entropy representation. This geometric formulation separates the advection, interaction, and diffusion processes occuring into separate operators and establishes the energy-preserving interconnections between them. Structural properties such as mass, momentum and energy conservation as well as entropy production arise naturally from the geometric formulation. As an application, we briefly discuss a framework for the energy control of the 3T system within the port-Hamiltonian framework.

1 Introduction

In this work, we present geometric formulations of the three-temperature radiation hydrodynamics (3T) system. First, we express the 3T system in the language of differential forms, both in an entropy representation and an energy representation. Subsequently, we show that the non-interacting and non-dissipative limiting system possesses a Lie–Poisson structure. Adding in interactions and dissipation yields an irreversible port-Hamiltonian structure.

There is an extensive literature on numerical methods for radiation hydrodynamics, see, for example, [1, 2, 3, 4, 5]. However, there is little work on the use of geometric numerical integration for radiation hydrodynamics. We establish the geometry of a radiation hydrodynamics system to enable future work on geometric numerical integration methods for radiation hydrodynamics problems.

1.1 The Three-temperature Radiation Hydrodynamics System

Radiation hydrodynamics models high energy-density plasmas of coupled electron, ion, and radiation fields; the electron and ion fields are described by fluid continuity equations and the radiation field is described by the radiation transport equation, coupled together through interaction terms (for details, see [6, 7, 8]). The 3T system can be viewed as a coarse-graining limit of full radiation hydrodynamics, where one combines the electron and ions into a single fluid (physically, this is the non-ionizing limit) and we take the electron, ion, and radiation energy densities to all be advected along this fluid. The coarse-graining is visualized schematically in Figure 1. The full radiation hydrodynamics system can be considered as three systems coupled through their interactions whereas the 3T system can be considered as a single self-interacting fluid.

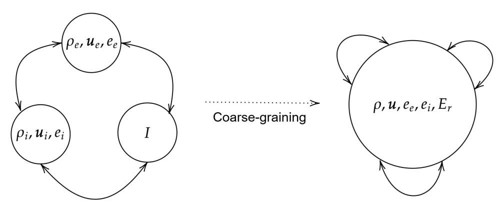

Figure 1: Coarse-graining of the full radiation hydrodynamics system to the 3T system.

The 3T equations take the form

$$\frac{\partial D}{\partial t} + \nabla \cdot (D\mathbf{u}) = 0, \tag{1a}$$

$$\frac{\partial}{\partial t} D\mathbf{u} + \nabla \cdot [D\mathbf{u} \otimes \mathbf{u} + (p_e + p_i + p_r)\mathbf{I}] = 0, \tag{1b}$$

$$\frac{\partial}{\partial t} De_i + \nabla \cdot [De_i\mathbf{u}] + \nabla \cdot \mathbf{F}_i = -p_i \nabla \cdot \mathbf{u} + G_{ei}(T_e, T_i) + S_i, \tag{1c}$$

$$\frac{\partial}{\partial t} De_e + \nabla \cdot [De_e\mathbf{u}] + \nabla \cdot \mathbf{F}_e = -p_e \nabla \cdot \mathbf{u} - G_{ei}(T_e, T_i) - G_{er}(T_e, T_r) + S_e, \tag{1d}$$

$$\frac{\partial}{\partial t} E_r + \nabla \cdot [E_r\mathbf{u}] + \nabla \cdot \mathbf{F}_r = -p_r \nabla \cdot \mathbf{u} + G_{er}(T_e, T_r) + S_r, \tag{1e}$$

where D and $\mathbf{u}$ are the material density scalar and velocity, e_e and e_i are the are the specific internal energies of electrons and ions, E_r is the radiation energy, T_e, T_i, T_r are the electron, ion, and radiation temperatures, respectively, p_e, p_i, p_r are the electron, ion and radiation pressures, respectively, $G_{\alpha\beta}$ denotes the energy-exchange coupling between species α and β $(\alpha,\beta=e,i,r)$, $\mathbf{F}_e,\mathbf{F}_i$ and $\mathbf{F}_r$ are the thermal and radiative fluxes, and S_ν are external source terms. For discussion of 3T radiation hydrodynamics, properties of its solutions, and numerical methods, see for example [1, 2, 3, 4, 7, 8].

To close the system, we define the temperatures, fluxes, interaction terms, and equations of state. The electron and ion temperatures can be related to their respective energies by an equation of state $T_\alpha = f_{T_\alpha}(\rho, e_\alpha),\ \alpha = i, e$. The thermal fluxes are given by $\mathbf{F}_\alpha = -K_\alpha(T_\alpha)\nabla T_\alpha,\ \alpha = e, i$, where K_α is the conduction coefficient, which is generally a positive nonlinear function of the temperature T_α. A standard choice is the Spitzer-Harm model $K_\alpha(T_\alpha) = d_\alpha T_\alpha^{5/2}$, for a fixed constant d_α [1]. There as several choices for the radiative flux. One possible choice is $\mathbf{F}_r = -\mathcal{D}(T_e)\nabla E_r$, where $\mathcal{D}$ is an electron temperature dependent radiative diffusion coefficient. This is specified by a complete equation of state (EOS) which also determines the pressures as functions of energy and density. Another possible choice is to take the second moment of the intensity transport equation to get an evolution equation for the radiative flux and close this system using a choice for the Eddington factor. The interaction terms take the form $G_{ei} = \kappa(T_e - T_i),\quad G_{er} = \sigma_p ac T_e^4 - \sigma_E c E_r$, where σ_E, σ_P are the opacities and κ is the electron-ion temperature relaxation coefficient, each a nonlinear function of the temperatures, specified by a complete EOS.

We now consider several properties of the 3T system.

Conservation Laws. Of course, (1a) and (1b) are the mass conservation and momentum conservation laws, respectively. To show energy conservation, we define the scalar energy density $E = \frac{1}{2}Du^2 + De_e + De_i + E_r$. Using (1a)-(1e), the time derivative of the total energy is given by

$$\frac{\partial E}{\partial t} + \nabla\cdot\left[(De_e + De_i + E_r)\mathbf{u} + \frac{1}{2}Du^2\mathbf{u} + p\mathbf{u} + \mathbf{F}_e + \mathbf{F}_i + \mathbf{F}_r\right] = \sum_{\nu=i,e,r} S_\nu, \tag{2}$$

where we defined the total pressure $p = p_e + p_i + p_r$. This is a local conservation law for the scalar energy density, modulo external sources S_ν.

Entropy Production. We derive the entropy production of the 3T system (1a)-(1e). In the remainder, we assume black body radiation so that $E_r = aT_r^4$ and $G_{er}(T_e,T_r) = \sigma_p ac(T_e^4 - T_r^4)$. Let s_α denote the specific entropy of species α and $s = s_e + s_i + s_r$ denote the total specify entropy; the Gibbs relations are given by

$$T_\alpha \frac{ds_\alpha}{dt} = \frac{de_\alpha}{dt} - \frac{p_\alpha}{D^2}\frac{dD}{dt},\ \alpha = e, i, \tag{3a}$$

$$T_r \frac{ds_r}{dt} = \frac{d}{dt}\left(\frac{E_r}{D}\right) - \frac{p_r}{D^2}\frac{dD}{dt}. \tag{3b}$$

Dividing each of the Gibbs relations by its respective temperature and summing the result, $D\frac{ds}{dt} = -\sum_{\alpha=e,i,r}\frac{1}{T_\alpha}\nabla\cdot\mathbf{F}_\alpha + \kappa\frac{(T_e-T_i)^2}{T_eT_i} + \frac{\sigma_p ac}{T_rT_e}(T_e - T_r)(T_e^4 - T_r^4) + \sum_\nu \frac{S_\nu}{T_\nu}$.

We write this in conservative form, using the definition of the material derivative, the continuity equation, and the identity $-T_\alpha^{-1}\nabla\cdot\mathbf{F}_\alpha = -\nabla\cdot(T_\alpha^{-1}\mathbf{F}_\alpha) - T_\alpha^{-2}\mathbf{F}_\alpha\cdot\nabla T_\alpha$. This yields the entropy production equation

$$\begin{aligned}\frac{\partial}{\partial t}Ds + \nabla\cdot\left(Ds\mathbf{u} + \frac{\mathbf{F}_e}{T_e} + \frac{\mathbf{F}_i}{T_i} + \frac{\mathbf{F}_r}{T_r}\right)& \\ = \frac{K_e(T_e)}{T_e^2}\|\nabla T_e\|^2 + \frac{K_i(T_i)}{T_i^2}\|\nabla T_i\|^2 + 4aT_rD(T_e)\|\nabla T_r\|^2 & \\ + \kappa\frac{(T_e - T_i)^2}{T_eT_i} + \frac{\sigma_p ac}{T_rT_e}(T_e - T_r)(T_e^4 - T_r^4) + \sum_\nu \frac{S_\nu}{T_\nu}.&\end{aligned} \tag{4}$$

Other than the external source terms S_ν, all of the terms on the right hand side are non-negative corresponding to internal entropy production.

1.2 The 3T System in the Language of Differential Forms

To develop a geometric description of the 3T system, it will first be useful to recast to the 3T system in the language of differential forms. For background on geometric formulations of hydrodynamics, we refer the reader to [9, 10]. Let a Riemannian manifold Q be the spatial domain and Ω be a fixed (constant-in-time) volume form on Q (we assume $\dim(Q) = 3$ for simplicity to fix sign conventions involving the Hodge star operator). We denote by $\boldsymbol{u}^\flat$ and $\boldsymbol{\omega}^\sharp$ the Riemannian lowering and raising operators, mapping tangent vectors to cotangent vectors and vice-versa, respectively and by $*$ the Hodge star operator.

The 3T system in the entropy representation. To begin, we start with the 3T system defined by the mass, momentum, and entropy evolution equations. The mass continuity equation $dD/dt = 0$ as the density scalar is advected. We introduce the density form $\rho = D\Omega$ which is a volume form on Q. Thus, the mass continuity equation can be rewritten $d\rho/dt = 0$. For the momentum equation, we introduce the 1–form density $\boldsymbol{P} = \boldsymbol{u}^\flat \otimes \rho = D\boldsymbol{u}^\flat \otimes \Omega$. For the entropy equations, we introduce the species$-\nu$ entropy density $\sigma_\nu = s_\nu\rho$. The mass continuity, momentum continuity, and the Gibbs relations can then be expressed

$$(\partial_t + \mathcal{L}_{\boldsymbol{u}})\rho = 0, \tag{5a}$$

$$(\partial_t + \mathcal{L}_{\boldsymbol{u}})\boldsymbol{P} = -dp\otimes\Omega + d\left(\frac{1}{2}|\boldsymbol{u}|^2\right)\otimes D\Omega, \tag{5b}$$

$$(\partial_t + \mathcal{L}_{\boldsymbol{u}})\sigma_i = -\frac{1}{T_i}d^*(K_i dT_i)\Omega + \frac{1}{T_i}G_{ei}(T_e,T_i)\Omega + \frac{S_i}{T_i}\Omega \tag{5c}$$

$$(\partial_t + \mathcal{L}_{\boldsymbol{u}})\sigma_e = -\frac{1}{T_e}d^*(K_e dT_e)\Omega - \frac{1}{T_e}G_{ei}(T_e,T_i)\Omega - \frac{1}{T_e}G_{er}(T_e,T_r)\Omega + \frac{S_e}{T_e}\Omega, \tag{5d}$$

$$(\partial_t + \mathcal{L}_{\boldsymbol{u}})\sigma_r = -\frac{1}{T_r}d^*(K_r dT_r)\Omega + \frac{1}{T_r}G_{er}(T_e,T_r)\Omega + \frac{S_r}{T_r}\Omega. \tag{5e}$$

We refer to eq. (5) as the 3T system in the *entropy representation*.

The 3T system in the energy representation. While the 3T system above is expressed in terms of entropy evolution equations (5d)-(5e), it can also equivalently be expressed in terms of energy evolution equations. Defining the species internal energy densities $\varepsilon_e = e_e\rho, \varepsilon_i = e_i\rho, \varepsilon_r = E_r\Omega$, the 3T system in the *energy representation* is

$$(\partial_t + \mathcal{L}_{\boldsymbol{u}})\rho = 0, \tag{6a}$$

$$(\partial_t + \mathcal{L}_{\boldsymbol{u}})\boldsymbol{P} = -dp \otimes \Omega + d\left(\frac{1}{2}|\boldsymbol{u}|^2\right) \otimes D\Omega, \tag{6b}$$

$$(\partial_t + \mathcal{L}_{\boldsymbol{u}})\varepsilon_i = -p_i d^*\mathbf{u}^\flat - d^*(K_i dT_i)\Omega + G_{ei}(T_e, T_i)\Omega + S_i\Omega, \tag{6c}$$

$$(\partial_t + \mathcal{L}_{\boldsymbol{u}})\varepsilon_e = -p_e d^*\mathbf{u}^\flat - d^*(K_e dT_e)\Omega - G_{ei}(T_e, T_i)\Omega - \frac{1}{T_e}G_{er}(T_e, T_r)\Omega + S_e\Omega, \tag{6d}$$

$$(\partial_t + \mathcal{L}_{\boldsymbol{u}})\varepsilon_r = -p_r d^*\mathbf{u}^\flat - d^*(K_r dT_r)\Omega + G_{er}(T_e, T_r)\Omega + S_r\Omega. \tag{6e}$$

2 Geometric Formulation of the 3T System

We first consider the case when dissipative effects and species interactions are neglected; then, the 3T system reduces to a Lie-Poisson Hamiltonian system on the dual to the Lie algebra $\mathfrak{g} = \mathfrak{X}(Q) \ltimes (C(Q) \times C(Q) \times C(Q) \times C(Q)) \ni (\boldsymbol{u}, g, g_i, g_e, g_r)$. The underlying Lie group $G = \mathrm{Diff}(Q) \ltimes (C(Q) \times C(Q) \times C(Q) \times C(Q)) \ni (\varphi, G, G_i, G_e, G_r)$ is the semidirect product of the diffeomorphism group $\mathrm{Diff}(Q) \ni \varphi$ with the abelian group $C(Q) \times C(Q) \times C(Q) \times C(Q) \ni (G, G_i, G_e, G_r)$. The diffeomorphisms $\mathrm{Diff}(Q)$ act from the left on $C(Q) \times C(Q) \times C(Q) \times C(Q)$ by pushforward. The group product in G is therefore $(\varphi^1, G^1, G^1_i, G^1_e, G^1_r)(\varphi^2, G^2, G^2_i, G^2_e, G^2_r) = (\varphi^1 \circ \varphi^2, G^1 + \varphi^1_* G^2, G^1_i + \varphi^1_* G^2_i, G^1_e + \varphi^1_* G^2_e, G^1_r + \varphi^1_* G^2_r)$. The corresponding Lie product on $\mathfrak{g}$ is

$$\left[\begin{pmatrix} \boldsymbol{u}^1 \\ g^1 \\ g^1_i \\ g^1_e \\ g^1_r \end{pmatrix}, \begin{pmatrix} \boldsymbol{u}^2 \\ g^2 \\ g^2_i \\ g^2_e \\ g^2_r \end{pmatrix}\right] = -\begin{pmatrix} [\boldsymbol{u}^1, \boldsymbol{u}^2] \\ \mathcal{L}_{\boldsymbol{u}^1} g^2 - \mathcal{L}_{\boldsymbol{u}^2} g^1 \\ \mathcal{L}_{\boldsymbol{u}^1} g^2_i - \mathcal{L}_{\boldsymbol{u}^2} g^1_i \\ \mathcal{L}_{\boldsymbol{u}^1} g^2_e - \mathcal{L}_{\boldsymbol{u}^2} g^1_e \\ \mathcal{L}_{\boldsymbol{u}^1} g^2_r - \mathcal{L}_{\boldsymbol{u}^2} g^1_r \end{pmatrix}. \tag{7}$$

The Lie-Poisson bracket between functionals $G, H : \mathfrak{g}^* \to \mathbb{R}$ on $\mathfrak{g}^*$, for $\xi^* = (\boldsymbol{P}, \rho, \sigma_i, \sigma_e, \sigma_r) \in \mathfrak{X}^*(Q) \times C^*(Q)^4$, is

$$\begin{aligned} \{G, H\}_{\mathfrak{g}^*}(\xi^*) &= \left\langle \xi^*, \left[\frac{\delta G}{\delta \xi^*}, \frac{\delta H}{\delta \xi^*}\right]\right\rangle \\ &= -\int \boldsymbol{P} \cdot \left[\frac{\delta G}{\delta \boldsymbol{P}}, \frac{\delta H}{\delta \boldsymbol{P}}\right] - \int \rho \left(\mathcal{L}_{\delta G/\delta \boldsymbol{P}} \frac{\delta H}{\delta \rho} - \mathcal{L}_{\delta H/\delta \boldsymbol{P}} \frac{\delta G}{\delta \rho}\right) - \int \sigma_i \left(\mathcal{L}_{\delta G/\delta \boldsymbol{P}} \frac{\delta H}{\delta \sigma_i} - \mathcal{L}_{\delta H/\delta \boldsymbol{P}} \frac{\delta G}{\delta \sigma_i}\right) \\ &\quad - \int \sigma_e \left(\mathcal{L}_{\delta G/\delta \boldsymbol{P}} \frac{\delta H}{\delta \sigma_e} - \mathcal{L}_{\delta H/\delta \boldsymbol{P}} \frac{\delta G}{\delta \sigma_e}\right) - \int \sigma_r \left(\mathcal{L}_{\delta G/\delta \boldsymbol{P}} \frac{\delta H}{\delta \sigma_r} - \mathcal{L}_{\delta H/\delta \boldsymbol{P}} \frac{\delta G}{\delta \sigma_r}\right). \end{aligned} \tag{8}$$

The Hamiltonian for the matter-radiation system can be expressed as the sum of kinetic and internal energies,

$$H(\xi^*) = \frac{1}{2}\int \left|\frac{\boldsymbol{P}}{\rho}\right|^2 \rho + \int \mathcal{U}_i\left(\frac{\rho}{\Omega}, \frac{\sigma_i}{\rho}\right)\rho + \int \mathcal{U}_e\left(\frac{\rho}{\Omega}, \frac{\sigma_e}{\rho}\right)\rho + \int \mathcal{U}_r\left(\frac{\rho}{\Omega}, \frac{\sigma_r}{\rho}\right)\rho, \tag{9}$$

where $\mathcal{U}_\nu$ denotes the species$-\nu$ specific internal energy, interpreted as a function of the density scalar $D = \rho/\Omega$ and the specific entropy $s_\nu = \sigma_\nu/\rho$. Its functional derivatives are given by

$$\frac{\delta H}{\delta \boldsymbol{P}} = \left(\frac{\boldsymbol{P}}{\rho}\right)^\sharp, \quad \frac{\delta H}{\delta \rho} = -\frac{1}{2}\left|\frac{\boldsymbol{P}}{\rho}\right|^2 + F_i + F_e + F_r, \quad \frac{\delta H}{\delta \sigma_i} = T_i, \quad \frac{\delta H}{\delta \sigma_e} = T_e, \quad \frac{\delta H}{\delta \sigma_r} = T_r,$$

where we have introduced specific Gibbs free energies and temperatures according to $F_\nu = \mathcal{U}_\nu + D\,\partial_D \mathcal{U}_\nu - s_\nu\,\partial_{s_\nu}\mathcal{U}_\nu$, $T_\nu = \partial_{s_\nu}\mathcal{U}_\nu$.

The non-interacting and non-dissipative limit of the 3T system can then be expressed as a Lie–Poisson system with bracket $\{\cdot,\cdot\}_{\mathfrak{g}^*}$ and Hamiltonian $\mathcal{H}$, with equations of motion

$$(\partial_t + \mathcal{L}_{\boldsymbol{u}})\boldsymbol{P} = -\frac{1}{D} d[p_i + p_e + p_r] \otimes \rho + d\left(\frac{1}{2}|\boldsymbol{u}|^2\right) \otimes \rho, \quad \boldsymbol{u} = \left(\frac{\boldsymbol{P}}{\rho}\right)^\sharp, \tag{10a}$$

$$(\partial_t + \mathcal{L}_{\boldsymbol{u}})\rho = 0, \quad (\partial_t + \mathcal{L}_{\boldsymbol{u}})\sigma_i = 0, \quad (\partial_t + \mathcal{L}_{\boldsymbol{u}})\sigma_e = 0, \quad (\partial_t + \mathcal{L}_{\boldsymbol{u}})\sigma_r = 0, \tag{10b}$$

where species-ν pressure is defined according to $p_\nu = D^2\,\partial_D \mathcal{U}_\nu$. These are precisely equations (5a)-(5e) with thermal flux terms, interaction terms, and sources set to zero.

2.1 Port-Hamiltonian Formulation of the 3T System

We now formulate the 3T system as a port-Hamiltonian system. For an overview of port-Hamiltonian systems, see [11]. For a discussion of port-Hamiltonian modelling of various fluid dynamical systems, see [12, 13, 14].

Define the space of flow variables $f \in \mathcal{F} = T\mathfrak{g}^*$ and the space of effort variables as its dual $\epsilon \in \mathcal{E} = T^*\mathfrak{g}^*$, given by $f_{\boldsymbol{P}} = -\frac{\partial \boldsymbol{P}}{\partial t}$, $\epsilon_{\boldsymbol{P}} = \frac{\delta H}{\delta \boldsymbol{P}} = \left(\frac{\boldsymbol{P}}{\rho}\right)^\sharp$, $f_\rho = -\frac{\partial \rho}{\partial t}$, $\epsilon_\rho = -\frac{1}{2}\left|\frac{\boldsymbol{P}}{\rho}\right|^2 + F_i + F_e + F_r$, $f_{\sigma_\mu} = -\frac{\partial \sigma_\mu}{\partial t}$, $\epsilon_{\sigma_\mu} = T_\mu$ $(\mu = i, e, r)$ with duality pairing $\langle f, \epsilon\rangle = \int_Q \left[f_{\boldsymbol{P}} \cdot \epsilon_{\boldsymbol{P}} + f_\rho \epsilon_\rho + \sum_\mu f_{\sigma_\mu}\epsilon_{\sigma_\mu}\right]$. Letting $x = (\boldsymbol{P}\ \rho\ \sigma_i\ \sigma_e\ \sigma_r)^T \in \mathfrak{g}^*$, the equations of motion can be expressed $\dot{x}(t) = \Pi(x)\frac{\delta H}{\delta x}$, where $\Pi(x)$ is the Poisson tensor associated to the above Lie-Poisson structure and $\dot{x} \coloneqq \partial x/\partial t$. Defining the Dirac structure $D \subset T\mathfrak{g}^* \times T\mathfrak{g}$ as the graph of the Poisson tensor $D = \text{graph}(\Pi)$, this is equivalent to $\left(\dot{x}(t), \frac{\delta H}{\delta x}\right) \in D$.

To incorporate thermal interactions between the three species, we define the interaction matrix, using the entropy evolution equations (5d)-(5e). For $x = (\boldsymbol{P}\ \rho\ \sigma_i\ \sigma_e\ \sigma_r)^T \in \mathfrak{g}^*$ and $\xi = (\boldsymbol{v}\ F\ T_i\ T_i\ T_r)^T \in \mathfrak{g}$,

$$G(x)\frac{\delta H}{\delta x} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{G_{ei}(T_e,T_r)}{T_e T_i}\Omega & 0 \\ 0 & 0 & -\frac{G_{ei}(T_e,T_i)}{T_e T_i}\Omega & 0 & -\frac{G_{er}(T_e,T_r)}{T_e T_r}\Omega \\ 0 & 0 & 0 & \frac{G_{er}(T_e,T_r)}{T_e T_r}\Omega & 0 \end{pmatrix} \begin{pmatrix} \delta H/\delta \boldsymbol{P} \\ \delta H/\delta \rho \\ \delta H/\delta \sigma_i \\ \delta H/\delta \sigma_e \\ \delta H/\delta \sigma_r \end{pmatrix}.$$

To incorporate the thermal fluxes, the thermal fluxes in the entropy evolution equation can be expressed as

$$-\frac{1}{T_i}\nabla\cdot\boldsymbol{F}_i = \frac{K_i(T_i)}{T_i^2}dT_i\wedge *d\frac{\delta H}{\delta\sigma_i} + *d^*(\frac{K_i(T_i)}{T_i^2}dT_i\frac{\delta H}{\delta\sigma_i}), \tag{11a}$$

$$-\frac{1}{T_e}\nabla\cdot\boldsymbol{F}_e = \frac{K_e(T_e)}{T_e^2}dT_e\wedge *\frac{\delta H}{\delta\sigma_e} + *d^*(\frac{K_e(T_e)}{T_e^2}dT_e\frac{\delta H}{\delta\sigma_e}), \tag{11b}$$

$$-\frac{1}{T_i}\nabla\cdot\boldsymbol{F}_r = \frac{K_r(T_e,T_r)}{T_r^2}dT_r\wedge *d\frac{\delta H}{\delta\sigma_r} + *d^*(\frac{K_r(T_e,T_r)}{T_r^2}dT_r\frac{\delta H}{\delta\sigma_r}), \tag{11c}$$

where recall $\delta H/\delta\sigma_\alpha = T_\alpha$. We express it in this form since we see directly that they are formally skew-adjoint in $\delta H/\delta x$. Define $F(x) = \text{diag}[0\ 0\ (11a)\ (11b)\ (11c)]$. Since G and F are formally skew-adjoint operators, the graph $D_{G,F} = \text{graph}(\Pi+G+F)$ defines a Dirac structure. Then, the complete 3T system can be expressed as a port-Hamiltonian system

$$\left(\dot{x},\frac{\delta H}{\delta x}\right)\in D_{G,F}(x),\quad \dot{x}(t) = (\Pi(x)+G(x)+F(x))\,\frac{\delta H}{\delta x}. \tag{12}$$

Boundary Ports. We now assume Q has boundary ∂Q. The boundary variables can be obtained by integrating by parts the energy balance equation. Ignoring for now the thermal flux terms, the boundary effort and flow variables associated to advection can be expressed

$$\begin{pmatrix} f_\partial \\ \epsilon_\partial^{\boldsymbol{P}} \\ \epsilon_\partial^{\rho} \\ \epsilon_\partial^{\sigma_i} \\ \epsilon_\partial^{\sigma_e} \\ \epsilon_\partial^{\sigma_r}\end{pmatrix} = \begin{pmatrix} dS\hat{n}\cdot & 0 & 0 & 0 & 0 \\ -D\boldsymbol{u}\cdot & 0 & 0 & 0 & 0 \\ 0 & -D & 0 & 0 & 0 \\ 0 & 0 & -Ds_i & 0 & 0 \\ 0 & 0 & 0 & -Ds_e & 0 \\ 0 & 0 & 0 & 0 & -Ds_r \end{pmatrix}\begin{pmatrix} \delta H/\delta\boldsymbol{P}|_{\partial Q} \\ \delta H/\delta\rho|_{\partial Q} \\ \delta H/\delta\sigma_i|_{\partial Q} \\ \delta H/\delta\sigma_e|_{\partial Q} \\ \delta H/\delta\sigma_r|_{\partial Q}\end{pmatrix}, \tag{13}$$

where $\hat{n}$ is the outward unit normal to ∂Q and dS is the boundary volume form corresponding to the trace of Ω. Note that the boundary flow variable $f_\partial = \delta H/\delta\boldsymbol{P}|_{\partial Q}\cdot\hat{n}dS$ is the boundary flow variable for each boundary effort variable. The boundary port variables corresponding to the thermal fluxes can be obtained from the boundary contribution of the thermal flux to the energy $\langle\delta H/\delta x, F(x)\delta H/\delta x\rangle$, which for $\nu = i,e,r$ are simply

$$f_{\nu\partial}^{therm} = \frac{K_\nu}{T_\nu}\nabla T_\nu|_{\partial Q}\cdot\hat{n},\quad \epsilon_{\nu\partial}^{therm} = \frac{\delta H}{\delta\sigma_\nu}|_{\partial Q} = T_\nu|_{\partial Q}. \tag{14}$$

A direct calculation shows that the so-defined boundary port variables yield the correct energy balance, i.e., equation (2).

Proposition 1. *Let* $(\dot{x},\delta H/\delta x)\in D_{G,F}$ *be the port-Hamiltonian system with boundary ports* $(f_\partial\ \epsilon_\partial^{\boldsymbol{P}}\ \epsilon_\partial^{\rho}\ \epsilon_\partial^{\sigma_i}\ \epsilon_\partial^{\sigma_e}\ \epsilon_\partial^{\sigma_r})$ *and* $(f_{\nu\partial}^{therm},\epsilon_{\nu\partial}^{therm})$ *defined by* (13) *and* (14), *then the energy balance is given by*

$$\frac{\partial}{\partial t}H = \sum_a\langle f_\partial,\epsilon_\partial^{x_a}\rangle_\partial + \sum_{\nu=i,e,r}\langle f_{\nu\partial}^{therm},\epsilon_{\nu\partial}^{therm}\rangle_\partial. \tag{15}$$

The 3T System as an Irreverisble Port-Hamiltonian System. We can view the above formulation of the 3T system as an irreversible port-Hamiltonian system [15, 16]. To establish this connection, we transport to the co-moving or Lagrangian frame, with $d/dt = \partial_t + \mathcal{L}_{\boldsymbol{u}}$ denoting the advective derivative. Letting $\boldsymbol{\sigma} := (\sigma_i\ \sigma_e\ \sigma_r)^T$, the 3T system can be expressed in block form

$$\frac{d}{dt}\left(\begin{array}{c}\boldsymbol{P}\\ \rho\\ \hline \boldsymbol{\sigma}\end{array}\right) = \left(\begin{array}{c|c}\Pi_0(x) & 0\\ \hline 0 & G_0(x)\end{array}\right)\left(\begin{array}{c}\delta H/\delta \boldsymbol{p}\\ \delta H/\delta\rho\\ \hline \delta H/\delta\boldsymbol{\sigma}\end{array}\right) + \left(\begin{array}{c|c}0 & 0\\ \hline 0 & F_0(x)\end{array}\right)\left(\begin{array}{c}\delta H/\delta \boldsymbol{p}\\ \delta H/\delta\rho\\ \hline \delta H/\delta\boldsymbol{\sigma}\end{array}\right), \tag{16}$$

where $\Pi_0(x)$ is the upper-left block of the Poisson tensor in the co-moving frame (the other blocks of the Poisson tensor vanish since the entropy is advected, (10), ignoring the thermal diffusion and interaction terms), and $G_0(x), F_0(x)$ are the 3×3 lower right blocks of $G(x)$ and $F(x)$, respectively.

This is essentially in the form of an "irreverisble Port-Hamiltonian system" as defined in [16], by adding in the evolution of the total entropy. Clearly, the species entropy production is encoded by G_0 and F_0 via $\frac{d}{dt}\boldsymbol{\sigma} = (G_0(x) + F_0(x))\frac{\delta H}{\delta\boldsymbol{\sigma}}$, We now compute the change in the total entropy $\mathcal{S} = \int_Q \sum_\nu \boldsymbol{\sigma}^\nu = \int_Q (s_i + s_e + s_r)\rho$. Let us further decompose F_0 into internal F_0^{in} and boundary F_0^{∂} components as

$$F_0^{in}(x)\frac{\delta H}{\delta\boldsymbol{\sigma}} := \begin{pmatrix} \frac{K_i(T_i)}{T_i^2} dT_i \wedge *d(\cdot) & 0 & 0\\ 0 & \frac{K_e(T_e)}{T_e^2} dT_e \wedge *d(\cdot) & 0\\ 0 & 0 & \frac{K_r(T_e,T_r)}{T_r^2} dT_r \wedge *d(\cdot)\end{pmatrix}\frac{\delta H}{\delta\boldsymbol{\sigma}}. \tag{17}$$

and $F_0^{\partial}(x) = F_0(x) - F_0^{in}(x)$. Then, the change in the total entropy is

$$\begin{aligned}\frac{\partial\mathcal{S}}{\partial t} &= -\int_{\partial Q}\rho(s_i + s_e + s_r)\boldsymbol{u}\cdot\hat{n}dS + \int_Q\sum_\nu\left(F_0^{\partial}(x)\frac{\delta H}{\delta\boldsymbol{\sigma}}\right)^\nu + \int_Q\sum_\nu\left((G_0(x) + F_0^{in}(x))\frac{\delta H}{\delta\boldsymbol{\sigma}}\right)^\nu\\ &= \sum_\nu\left\langle\frac{f_\partial}{T_\nu}, \epsilon_\partial^{\sigma_\nu}\right\rangle_\partial + \sum_\nu\left\langle\frac{f_{\nu\partial}^{therm}}{T_\nu}, \epsilon_{\nu\partial}^{therm}\right\rangle_\partial + \int_Q\sum_\nu\left((G_0(x) + F_0^{in}(x))\frac{\delta H}{\delta\boldsymbol{\sigma}}\right)^\nu.\end{aligned} \tag{18}$$

The first two terms on the right-hand side corresponding to entropy flow through the boundary due to advection and thermal flux, respectively. We have expressed this in terms of the boundary port variables to see that both terms arise from heat transfer through the boundary ($\delta S = \delta Q/T$). The third term on the right-hand side corresponds to irreversible internal entropy production produced by thermal interaction (G_0) and diffusion (F_0^{in}) in the interior of the domain, since it equals the non-negative terms of the right hand side of the entropy production equation (4). The entropy balance equation (18) is similar to the entropy balance derived in [16], decomposing into boundary terms corresponding to heat flow and internal entropy production terms corresponding to irreversible processes. The irreverisible port-Hamiltonian structure of the 3T system is summarized in Figure 2.

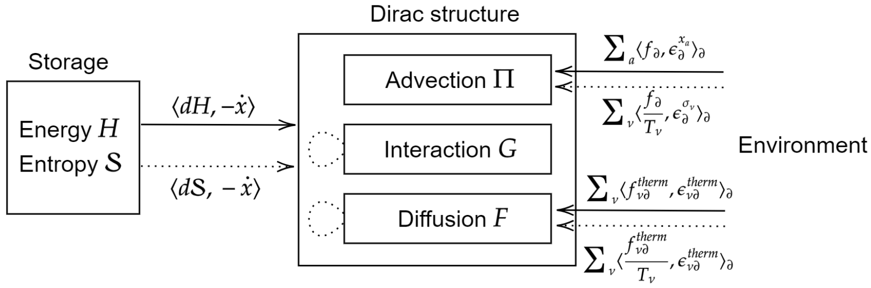

Figure 2: Schematic of the irreversible port-Hamiltonian structure of the 3T system. Solid lines indicate directions of energy flow. Dashed lines indicated directions of entropy flow, with the closed dashed loops indicating irreversible entropy production.

Application: Control of the 3T System. With the full port-Hamiltonian structure of the 3T system established, including boundary ports, it is straightforward to define control systems associated to the 3T system from the port-Hamiltonian framework. Consider the spatial domain Q without boundary and the 3T system with three heat sources S_i, S_e, S_r. Practically speaking, we may only want to provide the control input and measure the output in some localized region. Let Q_e, Q_i, Q_r be subsets of Q. We visualize this schematically in Figure 3.

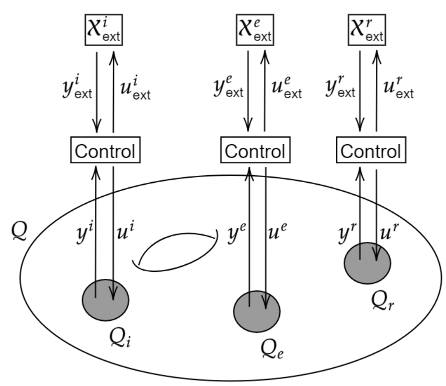

Figure 3: Schematic of the localized energy transfer control with sources compactly supported in the domain.

The 3T system with localized sources is given by equation (5), replacing the sources S_ν with the localized sources $S_\nu \varphi_\nu$, where φ_ν is a bump function on the domain Q_ν, $\nu = i, e, r$. We express this as an input-state-output system

$$\dot{x}(t) = (\Pi(x) + G(x) + F(x)) \frac{\delta H}{\delta x} + \mathcal{G}\mathbf{u}, \quad \mathbf{y}(t) = \mathcal{G}^* \frac{\delta H}{\delta x}, \tag{19}$$

where $\mathbf{u} = (0\ 0\ S_i/T_i\ S_e/T_e\ S_r/T_r)^T$ is the control input (source), $\mathcal{G} = \Omega\ \mathrm{diag}(0\ 0\ \varphi_i\ \varphi_e\ \varphi_r)$, $\mathcal{G}^*$ is the dual of $\mathcal{G}$ with respect to the pairing $\langle \alpha, \beta \rangle = \int_Q \alpha(*\beta)\Omega$ between 0–forms and $n = \dim(Q)$ forms, i.e., $\mathcal{G}^* = \mathrm{diag}(0\ 0\ \varphi_i\ \varphi_e\ \varphi_r)$ and $y = (0\ 0\ T_i\varphi_i\ T_e\varphi_e\ T_r\varphi_r)^T$ is the control output. The control inputs correspond to the localized sources and the control outputs correspond to the localized temperatures. Component-wise, we denote the non-zero control inputs as $u^\nu = S_\nu/T_\nu$ and the non-zero control outputs as $y^\nu = T_\nu\varphi_\nu$, $\nu = i, e, r$.

We will design the control inputs as to transfer energy into the 3T system from external systems or vice-versa. To do this, consider three additional external port-Hamiltonian systems, corresponding to each species (ions, electrons, and radiation),

$$x^\nu_{\mathrm{ext}} = J^\nu_{\mathrm{ext}}(x^\nu_{\mathrm{ext}}) \frac{\delta H^\nu_{\mathrm{ext}}}{\partial x^\nu_{\mathrm{ext}}}(x^\nu_{\mathrm{ext}}) + g^\nu_{\mathrm{ext}}(x^\nu_{\mathrm{ext}}) u^\nu_{\mathrm{ext}},\ \ \nu = i, e, r, \tag{20a}$$

$$y^\nu_{\mathrm{ext}} = g^\nu_{\mathrm{ext}}(x^\nu_{\mathrm{ext}})^* \frac{\delta H^\nu_{\mathrm{ext}}}{\partial x^\nu_{\mathrm{ext}}}(x^\nu_{\mathrm{ext}}),\ \ \nu = i, e, r, \tag{20b}$$

where each external port-Hamiltonian system has state space given by some Hilbert space $\mathcal{X}^\nu_{\mathrm{ext}}$, $u^\nu_{\mathrm{ext}} \in \mathbb{H}^\nu_{\mathrm{ext}}$ where $\mathbb{H}^\nu_{\mathrm{ext}}$ is a Hilbert space with inner product $(\cdot,\cdot)_{\mathbb{H}^\nu_{\mathrm{ext}}}$, $g^\nu_{\mathrm{ext}}(x^\nu_{\mathrm{ext}})$ is a linear mapping from $\mathcal{X}^\nu_{\mathrm{ext}}$ to $\mathbb{H}^\nu_{\mathrm{ext}}$ and $g^\nu_{\mathrm{ext}}(x^\nu_{\mathrm{ext}})^*$ is its adjoint.

Then, for each $\nu = i, e, r$, we define the control input and the external control input in terms of the outputs by a feedback interconnection

$$\begin{pmatrix} u^\nu \\ u^\nu_{\mathrm{ext}} \end{pmatrix} = \gamma_\nu \begin{pmatrix} T_\nu \|y^\nu_{\mathrm{ext}}\|^2_{\mathbb{H}^\nu_{\mathrm{ext}}} \\ -y^\nu_{\mathrm{ext}} \int_{Q_\nu} \varphi_\nu T_\nu^2 \Omega \end{pmatrix}, \tag{21}$$

for some chosen energy transfer coefficient $\gamma_\nu \neq 0$. In terms of the heat sources, this says to design the sources as $S_\nu = \gamma_\nu T_\nu u^\nu = \gamma_\nu T_\nu^2 \|y^\nu_{\mathrm{ext}}\|^2_{\mathbb{H}^\nu_{\mathrm{ext}}}$. Computing the change in the energy H for the 3T system as well as the external energy $H_{\mathrm{ext}} = \sum_\nu H^\nu_{\mathrm{ext}}$,

$$\dot{H} = \sum_\nu \langle y^\nu, u^\nu \rangle = \sum_\nu \gamma_\nu \int_{Q_\nu} \varphi_\nu T_\nu^2 \Omega \|y^\nu_{\mathrm{ext}}\|^2_{\mathbb{H}^\nu_{\mathrm{ext}}},$$

$$\dot{H}_{\mathrm{ext}} = \sum_\nu \gamma_\nu (y^\nu_{\mathrm{ext}}, u^\nu_{\mathrm{ext}})_{\mathbb{H}^\nu_{\mathrm{ext}}} = -\sum_\nu \int_{Q_\nu} \varphi_\nu T_\nu^2 \Omega \|y^\nu_{\mathrm{ext}}\|^2_{\mathbb{H}^\nu_{\mathrm{ext}}},$$

Note that the closed-loop Hamiltonian is conserved, $\dot{H} + \dot{H}_{\mathrm{ext}} = 0$ which corresponds to total energy conservation, due to the skew-symmetric structure of the output-to-input mapping, equation (21). If we choose each $\gamma_\nu > 0$, then $\dot{H} \geq 0$ and $\dot{H}_{\mathrm{ext}} \leq 0$, and vice-versa if each $\gamma_\nu < 0$. Furthermore, these inequalities can be made strict: assuming that the external systems are designed to be zero-state detectable, we will have strict inequalities since generally $T_\nu > 0$. Thus, we have constructed a feedback control for strict monotonic energy transfer between the 3T system and the external system.

3 Conclusion

We developed geometric formulations of three-temperature radiation hydrodynamics from an irreverisble port-Hamiltonian framework. This geometric formulation reveals the geometric structure in the individual processes occuring in the system, namely, advection, interaction, and diffusion, as well as the interconnection between these processes. Fundamental physical properties such as conservation and entropy production arise from the structural properties of the irreversible port-Hamiltonian framework. As an application, we provide a brief perspective on the energy control of the 3T system utilizing this formulation, which may be interesting in future applications for energy shaping and stabilization of radiation hydrodynamics systems. The geometric formulation of this system will enable the use of geometric structure-preserving integrators for the modelling, interconnection, and control in radiation hydrodynamics, which we will explore in future work.

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