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VECTOR POTENTIAL AND INTEGRAL REPRESENTATIONS OF SOLUTIONS TO CONSERVATIVE BOUNDARY VALUE PROBLEMS OF FLUID AND GAS DYNAMICS

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Abstract. Based on the created generalized apparatus of vector-tensor analysis, integral representations of the main dynamic and kinematic characteristics of the problem of viscous gas flow around force systems of arbitrary spatial shape are constructed. The boundary value problem of the interaction of such systems with a viscous gas flow is reduced to a system of linear, conditioned by physical boundary conditions, boundary integral equations regarding the kinematic and dynamic characteristics of the problem. It is proven that all the obtained characteristics depend on the newly obtained irrotational vector potential of the momentum, which significantly simplifies the integral representations of solutions and their numerical implementation.

On the basis of the created generalized apparatus of vector-tensor analysis, integral representations of the main dynamic and kinematic characteristics of the problem of the flow of a viscous gas flow around supporting systems of satisfactory spatial form have been constructed. The boundary value problem of the interaction of such systems with a viscous gas flow is reduced to a system of linear, conditioned by physical boundary conditions, boundary integral equations regarding the kinematic and dynamic characteristics of the problem. It is proven that all the obtained characteristics depend on the newly obtained, vortex-free vector potential of the momentum, which significantly simplifies the integral representations of the solutions and their numerical implementation.

INTRODUCTION

The main problem of continuum mechanics is the lack of correct methods for solving boundary value problems for systems of nonlinear partial differential equations, such as, for example, the system of Navier-Stokes equations of viscous gas dynamics. All existing attempts to create some kind of alternative mathematical models in differential forms have not yet been crowned with noticeable success; The arsenal of linearized problems has completely exhausted itself over the last century [1].

Currently, to solve current and popular problems of aero- hyrodynamics and gas dynamics, various methods for approximate solution of boundary value problems in the form of differential forms of mathematical models are widely used. Their common disadvantages are the bulkiness and unpredictability of results, high requirements for computing resources and, as a consequence, difficulties in solving optimization and economic feasibility problems [2]. It should be especially noted that to date, no qualitative methods have been developed for solving systems of nonlinear differential equations of the laws of conservation of fluid and gas mechanics. In addition, neither the existence nor the uniqueness of solutions to such systems can be proven [3], which raises many questions regarding the correspondence of the results obtained in this way to the physical processes and phenomena being studied. It is for this reason that a search is underway for models that allow the correct application of existing numerical analysis methods. Considerable efforts of modern researchers are aimed at constructing various differential forms of turbulence models, of which today there are more than fifty, and the search continues, although the results obtained leave much to be desired [3, 4].

Based on the ideology of the boundary integral equations (BIE) method, integrated computer technologies seem very promising [5]. The BIE method turned out to be most effective in cases of internal and external problems for unbounded areas with compact internal boundaries. It allows direct determination of distributed aero- hydrodynamic characteristics. Reducing a boundary value or initial boundary value problem to an integral equation or an adequate system of integral equations allows [5, 6]:

- reduce the dimension of the problem by one and consider more complex classes of problems than those solved by other methods;
- directly determine unknown quantities at the boundaries, without calculating them in the entire space of motion;
- determine the parameters of motion inside the region (pressure, vorticity, speed) by simple integration on the boundary of the region;
- reduce some hydrodynamic nonlinear problems for differential equations or systems of differential equations to a system of linear boundary integral equations with respect to unknown boundary values of parameters, desired problems or functions of them;
- pose and solve extreme problems that cannot be solved by any other method.

All this, of course, constitutes the advantages of the boundary integral equations method over finite difference methods and the finite element method. That is why this method is successfully used to solve complex engineering problems: plane and spatial, stationary and non-stationary [5].

The relevance of the issues is determined by the fact that, despite the increase in the number of computers and their productivity, the complexity and volume of tasks put forward by practice are ahead of progress in the development of computer technology. Consequently, the requirements for computational algorithms and, above all, for their efficiency, versatility and accuracy are increasing.

The mathematical model proposed in the work for the process of viscous gas flowing around a carrier system is a linear system of boundary integral equations, for which these issues are also resolved due to the assumed convergence of the computational process.

In connection with the above, the issue of generalization and extension of the method of boundary integral equations to boundary value problems of aerodynamics and gas dynamics with the construction of algorithmic foundations for the software for implementing this method is relevant.

1 GENERALIZED VECTOR-TENSOR ANALYSIS

The theoretical study of physical processes is based on the construction of corresponding mathematical models that do not contradict the physical laws. In continuum mechanics, the main mathematical apparatus is vector-tensor analysis. But along with the existing results, in this case there arises a natural need for a significant generalization of classical results in terms of both differential formulas and integral theorems.

1.1 Differential operations with vector-tensor functions

For further development of the method of boundary integral equations for the purpose of solving boundary value problems of the dynamics of a viscous gas, the following generalizations of the differential operations of vector-tensor analysis take place, proven in [6], in the case when the functions φ and the components of the vectors **A**, **B**, **G**, as well as tensors **G** and **Π** have the necessary differential properties in three-dimensional physical space.

First, it is necessary to introduce the standard tensor unit, which in a Cartesian basis has

$$\mathbf{I} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \mathbf{ii} + \mathbf{jj} + \mathbf{kk} \tag{1.1}$$

as well as tensor operators: gradient vector G

$$\nabla\mathbf{G} = \begin{Bmatrix} \dfrac{\partial G_x}{\partial x} & \dfrac{\partial G_y}{\partial x} & \dfrac{\partial G_z}{\partial x} \\ \dfrac{\partial G_x}{\partial y} & \dfrac{\partial G_y}{\partial y} & \dfrac{\partial G_z}{\partial y} \\ \dfrac{\partial G_x}{\partial z} & \dfrac{\partial G_y}{\partial z} & \dfrac{\partial G_z}{\partial z} \end{Bmatrix} = \mathbf{i}\frac{\partial \mathbf{G}}{\partial x} + \mathbf{j}\frac{\partial \mathbf{G}}{\partial y} + \mathbf{k}\frac{\partial \mathbf{G}}{\partial z}, \tag{1.2}$$

and its conjugate analogue

$$\nabla^*\mathbf{G} = \begin{Bmatrix} \dfrac{\partial G_x}{\partial x} & \dfrac{\partial G_x}{\partial y} & \dfrac{\partial G_x}{\partial z} \\ \dfrac{\partial G_y}{\partial x} & \dfrac{\partial G_y}{\partial y} & \dfrac{\partial G_y}{\partial z} \\ \dfrac{\partial G_z}{\partial x} & \dfrac{\partial G_z}{\partial y} & \dfrac{\partial G_z}{\partial z} \end{Bmatrix} = \mathbf{i}\nabla G_x + \mathbf{j}\nabla G_y + \mathbf{k}\nabla G_z \tag{1.3}$$

Then the following, either obvious or easily provable, but extremely necessary for further generalizations of differential formulas of vector-tensor analysis, take place:

$$(\nabla, (\mathbf{I}\varphi)) = \nabla\varphi; \tag{1.4$_1$}$$

$$[\nabla, \mathbf{I}\varphi] = [\mathbf{I}, \nabla\varphi]; \tag{1.4$_2$}$$

$$(\nabla, [\mathbf{I}, \mathbf{G}]) = [\nabla, \mathbf{G}]; \tag{1.4$_3$}$$

$$\left[\nabla, [\mathbf{I}, \mathbf{G}]\right] = \nabla^*\mathbf{G} - \mathbf{I}(\nabla, \mathbf{G}); \tag{1.4$_4$}$$

$$\left[\mathbf{I}, [\nabla, \mathbf{G}]\right] = \nabla^*\mathbf{G} - \nabla\mathbf{G}, \tag{1.4$_5$}$$

where

$$[\nabla,\mathbf{G}]=\mathbf{i}\left(\frac{\partial G_z}{\partial y}-\frac{\partial G_y}{\partial z}\right)+\mathbf{j}\left(\frac{\partial G_x}{\partial z}-\frac{\partial G_z}{\partial x}\right)+\mathbf{k}\left(\frac{\partial G_y}{\partial x}-\frac{\partial G_x}{\partial y}\right) \tag{1.5}$$

is ordinary vorticity vector.

Thus, the main differential formula of classical vector analysis takes the form:

$$\nabla(\nabla,\mathbf{G})=\Delta\mathbf{G}+\left[\nabla,\left[\mathbf{G}\right]\right]=\left(\nabla,\left\{\nabla\mathbf{G}+\nabla^*\mathbf{G}-\nabla\mathbf{G}\right\}\right)=(\nabla,\nabla^*\mathbf{G})\,. \tag{1.6$_1$}$$

And besides this, since $[\nabla,\nabla\mathbf{G}]=0$,

$$\left[\nabla,\left(\nabla^*\mathbf{G}\right)\right]=\left[\nabla,\left(\nabla^*\mathbf{G}-\nabla\mathbf{G}\right)\right]=\left[\nabla,\left[\mathrm{I},\nabla\varphi\right]\right]=\nabla\nabla\varphi. \tag{1.6$_2$}$$

For the purpose of further use, it is advisable to generalize some differential operations with two vector-tensor functions, which will differ somewhat from the classical ones due to the possible presence of tensor B:

$$\nabla\left(\mathbf{A},\mathbf{B}\right)=\left(\nabla\mathbf{A},\mathbf{B}\right)+\left(\mathbf{A},\nabla^*\mathbf{B}\right); \tag{1.7}$$

$$\begin{aligned}\left[\nabla,\left[\mathbf{A},\mathbf{B}\right]\right]&=\left(\left(\nabla^*\mathrm{A}\right),\mathbf{B}\right)-\left(\nabla,\mathbf{A}\right)\mathbf{B}-\left(\mathbf{A},\nabla\mathbf{B}\right)+\mathbf{A}\left(\nabla,\mathbf{B}\right)=\\&=\left(\left[\nabla,\left[\mathbf{I},\mathbf{A}\right]\right],\mathbf{B}\right)-\left(\mathbf{A},\left[\nabla,\left[\mathbf{I},\mathbf{B}\right]\right]\right)+\left[\mathbf{A},\left[\nabla,\mathbf{B}\right]\right],\end{aligned} \tag{1.8}$$

also for the purpose of further application in integral theorems from (1.8) we have:

$$\left(\mathbf{n},\left[\nabla,\left[\mathbf{A},\mathbf{B}\right]\right]\right)=\left(\mathbf{n},\left(\left[\nabla,\left[\mathbf{I},\mathbf{A}\right]\right],\mathbf{B}\right)\right)-\left(\mathbf{A},\left(\mathbf{n},\left(\left[\nabla,\left[\mathbf{I},\mathbf{B}\right]\right]\right)\right)\right); \tag{1.9}$$

$$\left[\mathbf{n},\nabla\left(\mathbf{A},\mathbf{B}\right)\right]=\left(\left[\mathbf{n},\nabla\mathbf{A}\right],\mathbf{B}\right)+\left(\left[\mathbf{n},\nabla\mathbf{B}^*\right],\mathbf{A}\right), \tag{1.10}$$

where $\mathbf{B}^* = \mathrm{B}_x\,\mathbf{i} + \mathrm{B}_y\,\mathbf{j} + \mathrm{B}_z\,\mathbf{k}$.

1.2 Integral theorems of generalized vector-tensor analysis

All known integral representations of solutions to classical boundary value problems of mathematical physics are based on integral theorems of vector analysis known as the Ostrogradsky-Gauss and Stokes-Green theorems.

First, the Ostrogradsky-Gauss theorem

$$\iiint\limits_{(E)}\left(\nabla,\mathbf{A}\right)dE=\oiint\limits_{(\partial E)}\left(\mathbf{n},\mathbf{A}\right)dS, \tag{1.11$_1$}$$

when using expressions (1.4), it can be constructed in several ways, taking into account algebraic operations with tensors:

$$\iiint\limits_{(E)}\nabla\varphi dE=\iiint\limits_{(E)}\left(\nabla,\mathbf{I}\varphi\right)dE=\oiint\limits_{(\partial E)}\mathbf{n}\varphi dS; \tag{1.11$_2$}$$

$$\iiint\limits_{(E)}\left[\nabla,\mathbf{A}\right]dE=\oiint\limits_{(\partial E)}\left[\mathbf{n},\mathbf{A}\right]dS; \tag{1.11$_3$}$$

$$\iiint\limits_{(E)}\nabla\left(\mathbf{A},\mathbf{B}\right)dE=\iiint\limits_{(E)}\left(\nabla,\mathbf{I}\left(\mathbf{A},\mathbf{B}\right)\right)dE=\oiint\limits_{(\partial E)}\mathbf{n}\left(\mathbf{A},\mathbf{B}\right)dS; \tag{1.11$_4$}$$

$$\iiint\limits_{(E)}[\nabla,[\mathbf{A},\mathbf{B}]]dE=\iiint\limits_{(E)}(\nabla,[\mathbf{I},(\mathbf{A},\mathbf{B})])dE=$$

$$=\oiint\limits_{(\partial E)}[\mathbf{n},[\mathbf{A},\mathbf{B}]]dS=\oiint\limits_{(\partial E)}\{\mathbf{A}(\mathbf{n},\mathbf{B})-(\mathbf{n},\mathbf{A})\mathbf{B}\}dS. \qquad (1.11_5)$$

And finally, we have the following generalized classical Ostrogradsky-Gauss theorems related to self-adjoint second-order differential operators:

$$\iiint\limits_{(E)}\{(\mathbf{B}^*,\nabla(\nabla,\mathbf{A}))-(\mathbf{A},\nabla(\nabla,\mathbf{B}))\}dE=\iiint\limits_{(E)}(\nabla,\{\mathbf{B}(\nabla,\mathbf{A})-\mathbf{A}(\nabla,\mathbf{B})\})dE=$$

$$=\oiint\limits_{(\partial E)}\{(\nabla,\mathbf{A})(\mathbf{n},\mathbf{B})-(\mathbf{n},\mathbf{A})(\nabla,\mathbf{B})\}dS. \qquad (1.12)$$

$$\iiint\limits_{(E)}\{([\nabla,[\nabla,\mathbf{A}]],\mathbf{B})-(\mathbf{A},[\nabla,[\nabla,\mathbf{B}]])\}dE=\iiint\limits_{(E)}(\nabla,\{[\mathbf{A},[\nabla,\mathbf{B}]]+[[\nabla,\mathbf{A}],\mathbf{B}]\})dE=$$

$$=\oiint\limits_{(\partial E)}\{([\mathbf{n},[\nabla,\mathbf{A}]],\mathbf{B})-(\mathbf{A},[\mathbf{n},[\nabla,\mathbf{B}]])\}dS, \qquad (1.13)$$

where tensor B* is the conjugate of tensor B.

Stokes formulas for vector-tensor expressions (1.7, 1.8) take the following form:

$$\iiint\limits_{(E)}[\nabla,\nabla(\mathbf{A},\mathbf{B})]dS=\oiint\limits_{(dE)}[\mathbf{n},\nabla(\mathbf{A},\mathbf{B})]dS=\oiint\limits_{(dE)}\{([\mathbf{n},\nabla\mathbf{A}],\mathbf{B})+([\mathbf{n},\nabla\mathbf{B}],\mathbf{A})\}dS; \qquad (1.14)$$

$$\oiint\limits_{(\partial E)}(\mathbf{n},[\nabla,[\mathbf{A},\mathbf{B}]])dS=\oiint\limits_{(\partial E)}\{((\mathbf{n},\{\nabla^*\mathbf{A}-\mathbf{I}(\nabla,\mathbf{A})\}),\mathbf{B})\}dS=$$

$$=\oiint\limits_{(\partial E)}\{((\mathbf{n},[\nabla,[\mathbf{I},\mathbf{A}]]),\mathbf{B})-((\mathbf{n},[\nabla,[\mathbf{I},\mathbf{B}]]),\mathbf{A})\}dS. \qquad (1.15)$$

1.3 Conservative forms of laws of conservation of viscous gas dynamics and vector momentum potential

Science has its own temples, on which many generations of our predecessors worked. Over the years, their branches have grown and reached canonical status; among them, there is also potential theory. The focus is on developing methods that can open the way to solving problems that were previously inaccessible.

Potential theory is a branch of mathematical physics that developed in connection with the theory of classical boundary conditions of mathematical physics (Laplace's equation, heat conduction, wave equation, and others). Obviously, the first essential step was related to the study of potential flows of an ideal incompressible fluid. The set of such currents turned out to be quite wide, and the mathematical possibilities of their research were almost infinite. However, all attempts to eliminate well-known paradoxes within the theory of an ideal incompressible fluid turned out to be fruitless, which proved the deficiency of this theory.

The most proven and tested mathematical model of fluid and gas mechanics is the system of conservation laws in differential form, which contain the law of conservation of mass or the equation of non-discontinuity

$$(\nabla, \rho\mathbf{V}) = 0; \quad (\mathbf{V}, \nabla\rho) + \rho(\nabla, \mathbf{V}) = 0; \tag{1.16}$$

and the law of conservation of momentum

$$(\nabla, \{\rho\mathbf{VV} + \mathbf{I}\mathrm{p} - \mathbf{T}\}) = 0, \tag{1.17}$$

where $\mathbf{T} = \{\tau_{ij}\}$ is the strain rate tensor, the elements of which are calculated according to the formulas:

$$\tau_{ii} = \frac{2\mu}{\mathrm{Re}}\frac{\partial v_i}{\partial x_i} - \frac{2\mu}{3\,\mathrm{Re}}(\nabla, \mathbf{V}); \quad \tau_{ij} = \frac{\mu}{\mathrm{Re}}\left(2\frac{\partial v_i}{\partial x_j} + \Omega_k\right); \quad \Omega_k = \frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}. \tag{1.18}$$

The system of equations (1.16 – 1.17) is also known as the system of Navier-Stokes equations [1 – 3]. It is advisable to supplement this system with the obvious law of conservation of vorticity:

$$(\nabla, \mathbf{\Omega}) = 0. \tag{1.19}$$

The momentum conservation law (1.17) reflects the conservatism of the momentum tensor

$$\mathbf{\Pi} = \rho\mathbf{VV} + \mathbf{I}\left\{p + \frac{2}{3}\frac{\mu}{\mathrm{Re}}(\nabla, \mathbf{V})\right\} - \frac{\mu}{\mathrm{Re}}\{2\nabla\mathbf{V} + [\mathbf{I}, \mathbf{\Omega}]\}, \tag{1.20}$$

which is naturally associated with the vector potential $\mathbf{\Psi}$

$$\begin{aligned} \mathbf{\Pi} &= [\nabla, [\mathbf{I}, \mathbf{\Psi}]] = \nabla^*\mathbf{\Psi} - \mathbf{I}(\nabla, \mathbf{\Psi}) = \\ &= \mathbf{i}\nabla\Psi_x + \mathbf{j}\nabla\Psi_y + \mathbf{k}\nabla\Psi_z - \mathbf{I}(\nabla, \mathbf{\Psi}), \end{aligned} \tag{1.21}$$

taking into account the expression (1.4_4).

The symmetry of the momentum tensor (1.20): $\Pi ij = \Pi ji$:

$$\begin{cases} \Pi_{xx} = \rho u^2 + p + \frac{2}{3}\frac{\mu}{\mathrm{Re}}(\nabla, \mathbf{V}) - 2\frac{\mu}{\mathrm{Re}}\frac{\partial u}{\partial x} = -\frac{\partial \Psi_y}{\partial y} - \frac{\partial \Psi_z}{\partial z}; \\ \Pi_{xy} = \rho uv - \frac{\mu}{\mathrm{Re}}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) = \frac{\partial \Psi_x}{\partial y}; \\ \Pi_{xz} = \rho uw - \frac{\mu}{\mathrm{Re}}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) = \frac{\partial \Psi_x}{\partial z}; \end{cases}$$

$$\begin{cases} \Pi_{yx} = \rho uv - \frac{\mu}{\mathrm{Re}}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) = \frac{\partial \Psi_y}{\partial x}; \\ \Pi_{yy} = \rho v^2 + p + \frac{2}{3}\frac{\mu}{\mathrm{Re}}(\nabla, \mathbf{V}) - 2\frac{\mu}{\mathrm{Re}}\frac{\partial v}{\partial y} = -\frac{\partial \Psi_x}{\partial x} - \frac{\partial \Psi_z}{\partial z}; \\ \Pi_{yz} = \rho vw - \frac{\mu}{\mathrm{Re}}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) = \frac{\partial \Psi_y}{\partial z}; \end{cases} \tag{1.22}$$

$$\begin{cases} \Pi_{zx} = \rho wu - \dfrac{\mu}{\mathrm{Re}}\left(\dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x}\right) = \dfrac{\partial \Psi_z}{\partial x}; \\ \Pi_{zy} = \rho wv - \dfrac{\mu}{\mathrm{Re}}\left(\dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial y}\right) = \dfrac{\partial \Psi_z}{\partial y}; \\ \Pi_{zz} = \rho w^2 + p + \dfrac{2}{3}\dfrac{\mu}{\mathrm{Re}}(\nabla, \mathbf{V}) - 2\dfrac{\mu}{\mathrm{Re}}\dfrac{\partial w}{\partial z} = -\dfrac{\partial \Psi_y}{\partial y} - \dfrac{\partial \Psi_z}{\partial z}; \end{cases}$$

proves the vorticity of the potential $\mathbf{\Psi}$:

$$\begin{gathered}\left[\mathbf{i}, \mathbf{\Pi}_x\right] + \left[\mathbf{j}, \mathbf{\Pi}_y\right] + \left[\mathbf{k}, \mathbf{\Pi}_z\right] = \\ = \mathbf{i}\left(\Pi_{yz} - \Pi_{zy}\right) + \mathbf{j}\left(\Pi_{zx} - \Pi_{xz}\right) + \mathbf{k}\left(\Pi_{xy} - \Pi_{yx}\right) = -\left[\nabla, \mathbf{\Psi}\right] = 0,\end{gathered} \quad (1.23)$$

and its conservatism (1.17), using (1.4_3)

$$\left(\nabla, \mathbf{\Pi}\right) = \Delta \mathbf{\Psi} - \nabla\left(\nabla, \mathbf{\Psi}\right) = -\left[\nabla, \left[\nabla, \mathbf{\Psi}\right]\right],$$

establishes, so to speak, the law of conservation of momentum potentiality

$$\left[\nabla, \left[\nabla, \mathbf{\Psi}\right]\right] = 0. \quad (1.24)$$

Here we also note that the expression

$$\left(\mathbf{i}, \mathbf{\Pi}_x\right) + \left(\mathbf{j}, \mathbf{\Pi}_y\right) + \left(\mathbf{k}, \mathbf{\Pi}_z\right) = \rho V^2 + 3p = -2\left(\nabla, \mathbf{\Psi}\right),$$

proves the following property of the vector potential Ψ:

$$\left(\nabla, \mathbf{\Psi}\right) = -\rho \frac{V^2}{2} - \frac{3}{2} p. \quad (1.25)$$

1.4 Fundamental solutions of differential operators

The construction of integral representations of solutions to boundary value problems of continuum mechanics using integral theorems of vector-tensor analysis is based on the use of fundamental solutions of the corresponding differential operators associated with the problem being solved.

Due to the fact that integral theorems (1.12, 1.13) transform volume integrals from a combination of second-order operators into surface ones, it is necessary to represent the conservation laws (1.16, 1.17, 1.19) in the same form: $\nabla\left(\nabla, \mathbf{A}\right) = 0,$, where

$$\mathbf{A} = \begin{cases} \rho \mathbf{V}; \\ \mathbf{\Pi}_i \; (i = x, y, z); \\ \mathbf{\Omega}. \end{cases} \quad (1.26)$$

The conservation law for the vector potential of momentum (1.24) is obtained in the form corresponding to the integral theorem (1.13).

Let us bring into consideration a tensor $\mathbf{\Gamma} = \mathrm{I}\varphi - \left[\mathrm{I}, \mathbf{G}\right]$, such that

$$\left(\nabla, \mathbf{\Gamma}\right) = 0 \Leftrightarrow \nabla \varphi = \left[\nabla, \mathbf{G}\right], \quad (1.27)$$

and

$$\left[\nabla,\mathbf{\Gamma}\right]=\left[\nabla,\mathrm{I}\varphi\right]-\left[\nabla,\left[\mathrm{I},\mathbf{G}\right]\right]=-\nabla\mathbf{G}+\mathrm{I}\left(\nabla,\mathbf{G}\right). \quad (1.28)$$

where φ is the known fundamental solution of the Laplace equation, and the vector G is precisely determined by equation (1.27). For example,

$$\mathbf{G}=\frac{cos\theta\nabla\omega}{4\pi}=\frac{z}{4\pi\sqrt{x^2+y^2+z^2}}\nabla\mathrm{arctg}\frac{y}{x}; \quad (1.29)$$

$$\left[\nabla,\mathbf{G}\right]==-\frac{\mathbf{r}}{4\pi r^3}=\nabla\left(\frac{1}{4\pi\left|\mathbf{x}-\mathbf{y}\right|}\right);\quad \left(\nabla,\mathbf{G}\right)=0.$$

Acting on the tensor Γ with the operator from (1.26), we successively obtain:

$$\nabla(\nabla,\ \mathbf{\Gamma}(|\mathbf{x}\text{-}\mathbf{y}|)) = \nabla(\nabla,\ \mathbf{I}\varphi) - \nabla(\nabla,\ [\mathbf{I},\ \boldsymbol{G}]) = \mathbf{I}\Delta\varphi - [\mathbf{I},\ \Delta\boldsymbol{G}] + [\nabla,\ (\mathbf{I}\ (\nabla,\ \boldsymbol{G}))] =$$

$$= \mathbf{I}\Delta\varphi - [\mathbf{I},\ \Delta\boldsymbol{G}] + [\mathbf{I},\ \nabla(\nabla,\ \boldsymbol{G})] = \mathbf{I}\Delta\varphi - [\mathbf{I}\ ,[\nabla,\ [\nabla,\ \boldsymbol{G}]] = \mathbf{I}\Delta\varphi + [\mathbf{I},\ [\nabla,\ (\nabla\varphi)]] = \mathbf{I}\delta\varphi; \quad (1.30)$$

i.e., the tensor Γ corresponds to the conditions for the existence of a fundamental solution of the operators of conservation of mass, momentum and vorticity.

As for the operator of conservation of the vector potential of momentum, we have:

$$\left[\nabla,\left[\nabla,\mathbf{\Gamma}\right]\right]=-\left[\nabla,\left\{\nabla\mathbf{G}-\mathbf{I}\left(\nabla,\mathbf{G}\right)\right\}\right]=0, \quad (1.31)$$

which proves that the tensor Γ is also fundamental in the case of operator (1.24), if we use the vector from (1.29), constructed specifically for this case.

Additionally, the Cartesian components of the tensor G are $\left(\nabla,\mathbf{G}\right)=0$:

$$\begin{aligned}\mathbf{\Gamma}_x&=\mathbf{i}\varphi+\mathbf{j}G_z-\mathbf{k}G_y;\\ \mathbf{\Gamma}_y&=-\mathbf{i}G_z+\mathbf{j}\varphi+\mathbf{k}G_x;\\ \mathbf{\Gamma}_x&=\mathbf{i}G_y-\mathbf{j}G_x+\mathbf{k}\varphi,\end{aligned} \quad (1.32)$$

has the following properties:

$$\left(\nabla,\mathbf{\Gamma}_x\right)=2\frac{\partial\varphi}{\partial x};\left(\nabla,\mathbf{\Gamma}_x\right)=2\frac{\partial\varphi}{\partial y};\left(\nabla,\mathbf{\Gamma}_x\right)=2\frac{\partial\varphi}{\partial z}; \quad (1.33)$$

$$\begin{aligned}\left[\nabla,\mathbf{\Gamma}_x\right]&=\frac{\partial\mathbf{G}}{\partial x}-\left[\mathbf{i},\nabla\varphi\right]=2\frac{\partial\mathbf{G}}{\partial x}-\nabla\mathbf{G}_x;\\ \left[\nabla,\mathbf{\Gamma}_y\right]&=\frac{\partial\mathbf{G}}{\partial y}-\left[\mathbf{j},\nabla\varphi\right]=2\frac{\partial\mathbf{G}}{\partial y}-\nabla\mathbf{G}_y;\\ \left[\nabla,\mathbf{\Gamma}_z\right]&=\frac{\partial\mathbf{G}}{\partial z}-\left[\mathbf{k},\nabla\varphi\right]=2\frac{\partial\mathbf{G}}{\partial z}-\nabla\mathbf{G}_z.\end{aligned} \quad (1.34)$$

2 STATEMENT OF THE BOUNDARY VALUE PROBLEM OF VISCOUS GAS FLOWING AROUND LOAD-BEARING AERODYNAMIC SYSTEMS OF ARBITRARY SPATIAL SHAPE.

The formulation of the boundary value problems of the flow of a viscous gas flow around supporting aerodynamic systems of satisfactory spatial form is formulated for the system of laws of conservation of mechanics of liquids and gases (1.16, 1.17) in the following expanded form:

$$(\nabla,\rho\mathbf{V})=(\mathbf{V},\nabla\rho)+\rho(\nabla,\mathbf{V})=0. \tag{2.1}$$

$$(\nabla,\mathbf{\Pi})=\rho(\mathbf{V},\nabla\mathbf{V})+\nabla p+\frac{\mu}{Re}[\nabla,\mathbf{\Omega}]=0;$$
$$(\nabla,\mathbf{\Pi}_x)=0;\quad(\nabla,\mathbf{\Pi}_y)=0;\quad(\nabla,\mathbf{\Pi}_z)=0, \tag{2.2}$$

where

$$\begin{cases}\mathbf{\Pi}_x=\rho u\mathbf{V}+\mathbf{i}\left\{p+\frac{2}{3}\frac{\mu}{\mathrm{Re}}(\nabla,\mathbf{V})\right\}-\frac{\mu}{\mathrm{Re}}\left\{2\frac{\partial\mathbf{V}}{\partial x}+[\mathbf{i},\mathbf{\Omega}]\right\}=-[\nabla,[\mathbf{i},\mathbf{\Psi}]];\\ \mathbf{\Pi}_y=\rho v\mathbf{V}+\mathbf{j}\left\{p+\frac{2}{3}\frac{\mu}{\mathrm{Re}}(\nabla,\mathbf{V})\right\}-\frac{\mu}{\mathrm{Re}}\left\{2\frac{\partial\mathbf{V}}{\partial y}+[\mathbf{j},\mathbf{\Omega}]\right\}=-[\nabla,[\mathbf{j},\mathbf{\Psi}]];\\ \mathbf{\Pi}_z=\rho w\mathbf{V}+\mathbf{k}\left\{p+\frac{2}{3}\frac{\mu}{\mathrm{Re}}(\nabla,\mathbf{V})\right\}-\frac{\mu}{\mathrm{Re}}\left\{2\frac{\partial\mathbf{V}}{\partial z}+[\mathbf{k},\mathbf{\Omega}]\right\}=-[\nabla,[\mathbf{k},\mathbf{\Psi}]]\end{cases} \tag{2.3}$$

conservative components of the momentum tensor (1.20), due to its symmetry ($\Pi_{ij}=\Pi_{ji}$).

The solution of this boundary value problem for the system of differential conservation equations (2.1, 2.2, 1.9) is correctly formulated by the method of boundary integral equations within the control volume (Fig. 2.1) with the location of the outer boundaries (Σ) at a sufficient distance from the supporting system, where the absence of disturbances in the viscous gas flow is guaranteed.

As proven by many years of physical experiments, the main boundary condition in the boundary conditions properties of the flow around solid, impenetrable supporting systems is non-slippage past the surface (S) of parts of the medium, i.e.

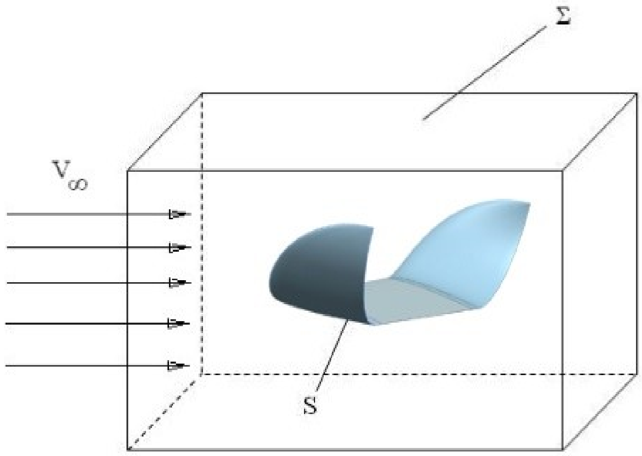

Fig. 2.1

$$\mathbf{V}\big|_{(S)}=\mathbf{V}(n)\big|_{(S)}=0. \tag{2.4}$$

and from (2.1) we have

$$(\nabla,\mathbf{V})\big|_{(S)}=0. \tag{2.5}$$

In order to determine the momentum tensor (1.17) on the surface (S) taking into account (2.4, 2.5), consider the vorticity vector Ω in the curvilinear orthogonal coordinate system. (s, τ, n):

$$\mathbf{\Omega}(s,\tau,n)\Big| = \frac{\mathbf{s}}{H_\tau H_n}\left\{\frac{\partial(H_n v_n)}{\partial\tau} - \frac{\partial(H_\tau v_\tau)}{\partial n}\right\} + \frac{\boldsymbol{\tau}}{H_n H_s}\left\{\frac{\partial(H_s v_s)}{\partial n} - \frac{\partial(H_n v_n)}{\partial s}\right\} +$$
$$+\frac{\mathbf{n}}{H_s H_\tau}\left\{\frac{\partial(H_\tau v_\tau)}{\partial s} - \frac{\partial(H_s v_s)}{\partial n}\right\}. \tag{2.6}$$

On a solid impervious surface (*S*), due to condition (2.4),

$$\mathbf{\Omega}\big|_{(S)}(s,\tau,n) = -\frac{\mathbf{s}}{H_n}\frac{\partial v_\tau}{\partial n} + \frac{\boldsymbol{\tau}}{H_n}\frac{\partial v_s}{\partial n}, \tag{2.7}$$

and the limit value has the form:

$$\mathbf{\Omega}_n\big|_{(S)}(s,\tau,n) = \frac{1}{H_s}\frac{\partial v_\tau}{\partial s} + \frac{1}{H_\tau}\frac{\partial v_s}{\partial \tau} = 0. \tag{2.8}$$

Thus, the components of the momentum tensor on the surface of the body in curvilinear coordinates have the form:

$$\begin{cases} \mathbf{\Pi}_s\big|_{(S)} = \mathbf{s}p - \dfrac{\mu}{Re}\dfrac{\mathbf{n}}{H_n}\dfrac{\partial v_s}{\partial n}; \\ \mathbf{\Pi}_\tau\big|_{(S)} = \boldsymbol{\tau}p - \dfrac{\mu}{Re}\dfrac{\mathbf{n}}{H_n}\dfrac{\partial v_\tau}{\partial n}; \\ \mathbf{\Pi}_n\big|_{(S)} = \mathbf{n}p - \dfrac{\mu}{Re}\dfrac{\mathbf{s}}{H_n}\dfrac{\partial v_s}{\partial n} - \dfrac{\mu}{Re}\dfrac{\boldsymbol{\tau}}{H_n}\dfrac{\partial v_\tau}{\partial n}, \end{cases} \tag{2.9}$$

and the Cartesian component, for example, Π_x is equal to:

$$\mathbf{\Pi}_x = (\mathbf{i},\mathbf{s})\mathbf{\Pi}_s + (\mathbf{i},\boldsymbol{\tau})\mathbf{\Pi}_\tau + (\mathbf{i},\mathbf{n})\mathbf{\Pi}_n. \tag{2.10}$$

3 INTEGRAL REPRESENTATIONS OF THE SOLUTIONS OF THE BOUNDARY VALUE PROBLEM OF THE FLOW OF A VISCOUS GAS AROUND THE SUPPORTING SYSTEM

The method of boundary integral equations has a number of unconditional advantages over finite-difference methods and the method of finite elements. That is why this method is currently successfully used to solve complex engineering problems: plane and spatial, stationary and time-dependent [6, 8].

The method of integral equations or the potential method for obtaining the solution of some partial differential equations is based on classical analysis. In recent years, this method has gained serious development as a result of its significant advantages over classical methods of numerical implementation of initial-boundary problems of mechanics, which are largely used by the aerospace complex of the developed countries of the world.

The essence of the method of boundary integral equations (BIE) for solving problems of mathematical physics is to reduce the boundary value problem for differential equations to an integral equation over the boundary of the domain, as a result of which the dimensionality of the problem is reduced by one and it becomes possible to solve more complex classes of problems than those solved by other methods. The advantage of the BIE method is also that it allows you to immediately determine unknown values at the boundaries, without calculating them over the entire area. Boundary value problems with the use of integral equations make it possible to solve more complex problems in both flat and three-dimensional formulation. Therefore, interest in potential theory is reviving again.

Unfortunately, in the existing packages of applied programs, the numerical solutions of the nonlinear system of the differential conservation laws of the mechanics of continuous media are based on the finite-difference and finite-element approaches [4], despite the fact that neither the existence nor the unity of the solutions have been proven to date connection of such systems [2]. The mathematical model of the viscous gas flow around the carrier system proposed in this study is represented by a linear system of boundary integral equations, for which these issues are also solved by the expected convergence of the computational process.

In connection with the above, the problem of generalization and distribution of the method of boundary integral equations in the boundary value problems of gas dynamics and the construction of algorithmic foundations of software tools for the implementation of this method is an urgent task.

3.1 Integral representations of solutions of differential equations of conservative conservation laws

In order to construct integral representations of solutions for equations of class (1.25), we use theorem (1.12) under the condition that $\mathbf{B} \equiv \mathbf{\Gamma} = \mathrm{I}\varphi - [\mathrm{I}, \mathbf{G}]$. Then

$$\iiint\limits_{(E)} \left\{ \left(\mathbf{\Gamma}^*, \nabla(\nabla, \mathbf{A}) \right) - \left(\mathbf{A}, \nabla(\nabla, \mathbf{\Gamma}) \right) \right\} dE = \oiint\limits_{(\partial E)} \left\{ (\nabla, \mathbf{A})(\mathbf{n}, \mathbf{\Gamma}) - (\mathbf{n}, \mathbf{A})(\nabla, \mathbf{\Gamma}) \right\} dS,$$

and after identical transformations, taking into account theorems (1.4), we obtain:

$$\oiint\limits_{(\partial E)} \left\{ \left(\left([\mathbf{n}, [\nabla, \mathbf{A}]] + \left(\frac{\partial \mathbf{A}}{\partial n} \right) \right), \mathbf{\Gamma} \right) - \left(\left(\mathbf{A}, [\mathbf{n}, [\nabla, \mathbf{\Gamma}]] \right) + \left(\mathbf{A}, \frac{\partial \mathbf{\Gamma}}{\partial n} \right) \right) \right\} dS = 0.$$

Paying attention to the fact that $\frac{\partial \mathbf{\Gamma}}{\partial n} = \mathbf{I} \frac{\partial \varphi}{\partial n} - \left[\mathbf{I}, \frac{\partial \mathbf{G}}{\partial n} \right]$, and using the known properties of the double layer potential $\frac{\partial \varphi}{\partial n}$, after the limit transition to the points of the surfaces, we have an integral representation of the solution of the equations of the conservative laws (1.25):

$$\mathbf{A} = \oiint\limits_{(\partial E)} \left\{ \left(\left([\mathbf{n}, [\nabla, \mathbf{A}]] + \left(\frac{\partial \mathbf{A}}{\partial n} \right) \right), \mathbf{\Gamma} \right) - \left(\left(\mathbf{A}, [\mathbf{n}, [\nabla, \mathbf{\Gamma}]] \right) + \left(\mathbf{A}, \frac{\partial \mathbf{\Gamma}}{\partial n} \right) \right) \right\} dS. \quad (3.1)$$

For further correct use of representation (3.1), we prove its conservativeness:

$$\left(\nabla_x, \mathbf{A}(\mathbf{x})\right) = \oiint\limits_{(S)} \left(\nabla_x, \left\{\left(\left(\left[\mathbf{n}, \left[\nabla, \mathbf{A}(\mathbf{y})\right]\right] + \left(\frac{\partial \mathbf{A}(\mathbf{y})}{\partial n}\right)\right), \mathbf{\Gamma}\left(\left|\mathrm{x} - \mathbf{y}\right|\right)\right) - \right.\right.$$
$$\left.\left. - \left(\mathbf{A}(\mathbf{y}), \left\{\left[\mathbf{n}, \left[\nabla, \mathbf{\Gamma}\left(\left|\mathrm{x} - \mathbf{y}\right|\right)\right]\right] + \frac{\partial \mathbf{\Gamma}\left(\left|\mathrm{x} - \mathbf{y}\right|\right)}{\partial n}\right\}\right)\right\}\right) dS. \tag{3.2}$$

Here is the differentiation of the integral

$$\oiint\limits_{(S)} \left(\left(\mathbf{n}, \nabla^* \mathbf{A}(\mathbf{y}), \right), \mathbf{\Gamma}\left(\left|\mathbf{x} - \mathbf{y}\right|\right)\right) dS$$

by the external variable x (see (1.32))

$$\left(\nabla_{\mathrm{x}}, \left(\left(\mathbf{n}, \nabla^* \mathbf{A}(\mathbf{y})\right), \mathbf{\Gamma}\left(\left|x - \mathbf{y}\right|\right)\right) = 2\left(\left(\mathbf{n}, \nabla^* \mathbf{A}(\mathbf{y})\right), \nabla\varphi\right),$$

and

$$\iint\limits_{(S)} \left(\mathbf{n}, \left[\nabla, \left[\mathbf{A}, \nabla\varphi\right]\right]\right) dS = \iint\limits_{(S)} \left\{\left(\left(\mathbf{n}, \nabla^* \mathrm{A}\right), \nabla\varphi\right) - \left(\nabla, \mathbf{A}\right)\frac{\partial\varphi}{\partial n} - \left(\mathbf{n}, \left(\mathbf{A}, \nabla\nabla\varphi\right)\right) + \left(\mathbf{n}, \mathbf{A}\right)\Delta\varphi\right\} dS;$$

$$\iint\limits_{(S)} \left(\mathbf{n}, \left(\mathbf{A}, \nabla\nabla\varphi\right)\right) dS = \iint\limits_{(S)} \left(\mathbf{A}, \left(\mathbf{n}, \nabla\nabla\varphi\right)\right) dS = \iint\limits_{(S)} \left(\mathbf{A}, \left(\mathbf{n}, \left[\nabla, \left[\mathbf{I}, \nabla\varphi\right]\right]\right)\right) dS =$$
$$= \iint\limits_{(S)} \left(\left(\mathbf{n}, \left[\nabla, \left[\mathbf{I}, \mathbf{A}\right]\right]\right), \nabla\varphi\right) dS = \iint\limits_{(S)} \left(\left(\mathbf{n}, \nabla^* \mathbf{A}\right), \nabla\varphi\right) dS.$$

after integration by parts, thanks to the conservativeness of the vector **A**, disappears, and

$$\left(\nabla_x, \left(\mathbf{A}(\mathbf{y}), \frac{\partial \mathbf{\Gamma}\left(\left|\mathbf{x} - \mathbf{y}\right|\right)}{\partial n}\right)\right) = \left(\mathbf{A}(\mathbf{y}), \frac{\partial \nabla\varphi}{\partial n}\right) + \left(\mathbf{A}(\mathbf{y}), \frac{\partial\left[\nabla, \mathbf{G}\right]}{\partial n}\right) = 2\left(\mathbf{A}(\mathbf{y}), \left(\mathbf{n}, \nabla\nabla\varphi\right)\right).$$

Thus, it is proven that the integral representation (3.1) is a solution to the boundary value problem for the class of equations (1.26) corresponding to the conservation laws (2.1, 2.2).

The obtained representation (3.1) makes it possible to write an integral representation of the solution of the law of conservation of mass (1.15):

$$\mathbf{V} = \oiint\limits_{(\partial E)} \left\{\left(\left(\left[\mathbf{n}, \left[\nabla, \mathbf{V}\right]\right] + \left(\frac{\partial \mathbf{V}}{\partial n}\right)\right), \mathbf{\Gamma}\right) - \left(\left(\mathbf{V}, \left[\mathbf{n}, \left[\nabla, \mathbf{\Gamma}\right]\right]\right) + \left(\mathbf{V}, \frac{\partial \mathbf{\Gamma}}{\partial n}\right)\right)\right\} dS, \tag{3.3}$$

where (see (1.19) and conditions (2.4, 2.5, 2.7))

$$\left[\mathbf{n}, \left[\nabla, \mathbf{V}\right]\right] + \frac{\partial \mathbf{V}}{\partial \mathrm{n}} = \frac{\mathrm{Re}}{\mu}\left(\mathbf{n}\left\{\mathbf{I}p - \left[\nabla, \left[\mathbf{I}, \mathbf{\Psi}\right]\right]\right\}\right) - \left[\mathbf{n}, \Omega_s\right]. \tag{3.4}$$

Besides,

$$\oiint\limits_{(\partial E)} \left(\mathbf{V}, \left[\mathbf{n}, \left[\nabla, \mathbf{\Gamma}\right]\right]\right) dS = \oiint\limits_{(\partial E)} \left(\mathbf{n}, \left[\mathbf{V}, \nabla \mathbf{G}\right]\right) dS = \oiint\limits_{(\partial E)} \left(\mathbf{n}, \left[\nabla, \mathbf{V}\right]\right) \mathbf{G} dS = 0.$$

The term $\left(\mathbf{n}, \left[\nabla, \left[\mathbf{I}, \mathbf{\Psi}\right]\right]\right)$ can be integrated by parts:

$$\oiint_{(S)}\left(\left(\mathbf{n},\left[\nabla,\left[\mathbf{I},\mathbf{\Psi}\right]\right]\right),\mathbf{\Gamma}\right)dS=\oiint_{(S)}\left(\left(\mathbf{n},\left[\nabla,\left[\mathbf{I},\mathbf{\Psi}\right]\right]\right),\left\{\mathbf{I}\varphi-\left[\mathbf{I},\mathbf{G}\right]\right\}\right)dS;$$

$$\oiint_{(S)}\left(\mathbf{n},\left[\nabla,\left[\mathbf{I},\mathbf{\Psi}\right]\right]\right)\varphi dS=\oiint_{(S)}\left[\mathbf{\Psi},\left[\mathbf{n},\nabla\varphi\right]\right]dS;$$

$$\oiint_{(S)}\left[\left(\mathbf{n},\left[\nabla,\left[\mathbf{I},\mathbf{\Psi}\right]\right]\right),\mathbf{G}\right]dS=-\oiint_{(S)}\left\{\left(\mathbf{\Psi},\left[\mathbf{n},\nabla\mathbf{G}\right]\right)-\left[\mathbf{n},\left(\mathbf{\Psi},\nabla^{*}\mathbf{G}\right)\right]\right\};$$

$$\oiint_{(S)}\left(\left(\mathbf{n},\left[\nabla,\left[\mathbf{I},\mathbf{\Psi}\right]\right]\right),\mathbf{\Gamma}\right)dS=\oiint_{(S)}\left\{\left[\mathbf{\Psi},\left[\mathbf{n},\nabla\varphi\right]\right]+\left(\mathbf{\Psi},\left[\mathbf{n},\nabla\mathbf{G}\right]\right)-\left[\mathbf{n},\left(\mathbf{\Psi},\nabla^{*}\mathbf{G}\right)\right]\right\}dS.$$

Thus, representation (3.2) acquires its final form

$$\mathbf{V}=\oiint_{(\partial E)}\left\{\left(\frac{\mathrm{Re}}{\mu}\left(\mathbf{n}p-\left[\mathbf{n},\mathbf{\Omega}_{s}\right]\right),\mathbf{\Gamma}\right)-\right.$$
$$\left.-\left\{\left[\mathbf{\Psi},\left[\mathbf{n},\nabla\varphi\right]\right]+\left(\mathbf{\Psi},\left[\mathbf{n},\nabla\mathbf{G}\right]\right)-\left[\mathbf{n},\left(\mathbf{\Psi},\nabla^{*}\mathbf{G}\right)\right]\right\}-\left(\mathbf{V},\frac{\partial\mathbf{\Gamma}}{\partial n}\right)\right\}dS. \qquad (3.5)$$

If we use representation (3.1) with respect to the vorticity vector Ω, we obtain

$$\mathbf{\Omega}=\oiint_{(\partial E)}\left\{\left(\left(\left[\mathbf{n},\left[\nabla,\mathbf{\Omega}\right]\right]+\left(\frac{\partial\mathbf{\Omega}}{\partial n}\right)\right),\mathbf{\Gamma}\right)-\left(\left(\mathbf{\Omega},\left[\mathbf{n},\left[\nabla,\mathbf{\Gamma}\right]\right]\right)+\left(\mathbf{\Omega},\frac{\partial\mathbf{\Gamma}}{\partial n}\right)\right)\right\}dS. \qquad (3.6)$$

The connection of the vorticity vector Ω with the vector potential of the momentum tensor Π allows rotation of the momentum tensor (1.20)

$$\left[\nabla,\mathbf{\Pi}\right]=\left[\nabla,\rho\mathbf{V}\mathbf{V}\right]+\left[\mathbf{I},\nabla\left\{p+\frac{2}{3}\frac{\mu}{\mathrm{Re}}\left(\nabla,\mathbf{V}\right)\right\}\right]-\frac{\mu}{\mathrm{Re}}\left\{\left[\mathbf{I},\left[\nabla,\mathbf{\Omega}\right]\right]+\nabla\mathbf{\Omega}\right\}=\left[\nabla,\left[\nabla,\left[\mathbf{I},\mathbf{\Psi}\right]\right]\right]. \qquad (3.7)$$

which leads to

$$\left[\mathbf{n},\left[\nabla,\mathbf{\Omega}\right]\right]+\left(\frac{\partial\mathbf{\Omega}}{\partial n}\right)=\left[\mathbf{n},\nabla\left\{\frac{Re}{\mu}p+\frac{2}{3}\left(\nabla,\mathbf{V}\right)\right\}\right]+$$
$$+\left(\mathbf{n},\left[\nabla,\frac{Re}{\nu}\mathbf{V}\mathbf{V}\right]\right)-\frac{Re}{\mu}\left(\mathbf{n},\left[\nabla,\left[\nabla,\left[\mathbf{I},\Psi\right]\right]\right]\right); \qquad (3.8)$$

$$\left[\mathbf{n},\left[\nabla,\mathbf{\Omega}\right]\right]+\left(\frac{\partial\mathbf{\Omega}}{\partial n}\right)\Bigg|_{(S)}=\frac{Re}{\mu}\left[\mathbf{n},\nabla p\right]+\frac{Re}{\mu}\left(\left[\mathbf{n},\left[\nabla\left(\nabla,\mathbf{\Psi}\right)\right]\right]\right)=\frac{Re}{\mu}\left[\mathbf{n},\nabla\left\{p-\left(\nabla,\mathbf{\Psi}\right)\right\}\right].$$

After appropriate integration in (3.6) by parts:

$$\oiint_{(\partial E)}\left(\mathbf{\Omega},\left[\mathbf{n},\left[\nabla,\mathbf{\Gamma}\right]\right]\right)dS=-\oiint_{(\partial E)}\left(\mathbf{\Omega},\left[\mathbf{n},\nabla\mathbf{G}\right]\right)dS=$$
$$=\oiint_{(\partial E)}\left(\mathbf{n},\left[\nabla,\left(\mathbf{\Omega}\mathbf{G}\right)\right]\right)dS-\oiint_{(\partial E)}\left(\mathbf{n},\left[\nabla,\mathbf{\Omega}\right]\right)\mathbf{G}dS=-\oiint_{(\partial E)}\frac{Re}{\mu}\frac{\partial p}{\partial n}\mathbf{G}dS;$$

$$\oiint_{(S)} \left(\left\{ \left[\mathbf{n},\left[\nabla,\mathbf{\Omega}\right]\right] + \left(\frac{\partial \mathbf{\Omega}}{\partial n}\right) \right\}, \mathbf{\Gamma} \right) dS = \frac{Re}{\mu} \oiint_{(S)} \left\{ \left(\mathbf{n},\nabla^{*}\mathbf{G}\right) - \left[\mathbf{n},\nabla\varphi\right] \right\} \left\{ p - \left(\nabla,\mathbf{\Psi}\right) \right\} dS,$$

we obtain the final expression of the integral representation of the vorticity vector **Ω**:

$$\mathbf{\Omega} = \frac{Re}{\mu} \oiint_{(\partial E)} \left\{ \left\{ \left(\mathbf{n},\nabla^{*}\mathbf{G}\right) - \left[\mathbf{n},\nabla\varphi\right] \right\} \left\{ p - \left(\nabla,\mathbf{\Psi}\right) \right\} + \frac{\mu}{Re}\frac{\partial p}{\partial n} - \left(\mathbf{\Omega},\frac{\partial \mathbf{\Gamma}}{\partial n}\right) \right\} dS. \tag{3.9}$$

Finally, based on sample (3.1), we construct the integral representation of the momentum tensor Π (1.20), which can be naturally formulated for the vector components Π_i ($i = x, y, z$) (2.3)

$$\mathbf{\Pi}_i = \oiint_{(\partial E)} \left\{ \left(\left(\left[\mathbf{n},\left[\nabla,\mathbf{\Pi}_i\right]\right] + \left(\frac{\partial \mathbf{\Pi}_i}{\partial n}\right) \right), \mathbf{\Gamma} \right) - \left(\left(\mathbf{\Pi}_i,\left[\mathbf{n},\left[\nabla,\mathbf{\Gamma}\right]\right]\right) + \left(\mathbf{\Pi}_i,\frac{\partial \mathbf{\Gamma}}{\partial n}\right) \right) \right\} dS. \tag{3.10}$$

Here, too, there is a connection with the momentum potential vector (2.3), which allows performing the transformation using integration by parts and obtain the final result, for example, for the vector $\mathbf{\Pi}_x$:

$$\mathbf{\Pi}_x = -\oiint_{(\partial E)} \left\{ \left(\left(\mathbf{n},\left[\nabla,\left[\mathbf{I},\left[\nabla\left[\mathbf{i},\mathbf{\Psi}\right]\right]\right]\right]\right), \mathbf{\Gamma} \right) - \left(\left(\mathbf{\Pi}_x,\left[\mathbf{n},\left[\nabla,\mathbf{\Gamma}\right]\right]\right) + \left(\mathbf{\Pi}_x,\frac{\partial \mathbf{\Gamma}}{\partial n}\right) \right) \right\} dS. \tag{3.11}$$

To integrate the first component in (3.11) by parts, we use a variant of the Stokes formula, where (**AB**) is a tensor dyad:

$$\oiint_{(\partial E)} \left(\mathbf{n},\left[\nabla,\left(\mathbf{AB}\right)\right]\right) dS = \oiint_{(\partial E)} \left\{ \left(\left(\mathbf{n},\left[\nabla,\mathbf{A}\right]\right),\mathbf{B}\right) - \left(\mathbf{n},\left[\mathbf{A},\nabla\mathbf{B}\right]\right) \right\} dS = 0. \tag{3.12}$$

Then the integral in (3.11) $\oiint_{(\partial E)} \left(\mathbf{\Pi}_x,\left[\mathbf{n},\left[\nabla,\mathbf{\Gamma}\right]\right]\right) dS$ is written as

$$\oiint_{(\partial E)} \left(\mathbf{\Pi}_x,\left[\mathbf{n},\left[\nabla,\mathbf{\Gamma}\right]\right]\right) dS = \oiint_{(\partial E)} \left(\mathbf{n},\left[\mathbf{\Pi}_x,\left[\nabla\mathbf{G}\right]\right]\right) dS = \oiint_{(\partial E)} \left(\left(\mathbf{n},\left[\nabla,\mathbf{\Pi}_x\right]\right),\mathbf{G}\right) dS.$$

After obvious transformations, we obtain

$$\begin{aligned} \mathbf{\Pi}_x = \oiint_{(\partial E)} \Big\{ & \left(\mathbf{n},\left\{ \left[\left(\mathbf{A},\mathbf{i}\right),\nabla\mathbf{\Gamma}_x\right] + \left[\left(\mathbf{A},\mathbf{j}\right),\nabla\mathbf{\Gamma}_y\right] + \left[\left(\mathbf{A},\mathbf{k}\right),\nabla\mathbf{\Gamma}_z\right] \right\}\right) - \\ & - \left(\left(\left(\mathbf{n},\left[\nabla,\mathbf{\Pi}_x\right]\right),\mathbf{G}\right) + \left(\mathbf{\Pi}_x,\frac{\partial \mathbf{\Gamma}}{\partial n}\right) \right) \Big\} dS, \end{aligned} \tag{3.13}$$

where $\mathbf{A} = \left[\mathbf{I},\left[\nabla\left[\mathbf{i},\mathbf{\Psi}\right]\right]\right]$

In order to construct an integral representation of the vector potential **Ψ**, we use the Stokes theorem (1.13)

$$\oiint_{(\partial E)} \left\{ \left(\left[\mathbf{n},\left[\nabla,\mathbf{\Psi}\right]\right],\mathbf{\Gamma}\right) - \left(\mathbf{\Psi},\left[\mathbf{n},\left[\nabla,\mathbf{\Gamma}\right]\right]\right) \right\} dS = 0,$$

whence after the limit transition, we have:

$$\mathbf{\Psi} = \oiint\limits_{(\partial E)} \left\{ \left(\left[\mathbf{n}, \left[\nabla, \mathbf{\Psi} \right] \right], \mathbf{\Gamma} \right) - \left(\mathbf{\Psi}, \left[\mathbf{n}, \left[\nabla, \mathbf{\Gamma} \right] \right] \right) \right\} dS. \tag{3.14}$$

Performing the transformation according to (1.43, 1.44), integration by parts according to the rule (1.9) and the connection of the vector potential with the momentum tensor **Π** (1.21), we obtain:

$$\mathbf{\Psi} = \oiint\limits_{(\partial E)} \left\{ \left(\left(\mathbf{n}, \mathbf{\Pi} \right), \mathbf{\Gamma} \right) - \left(\mathbf{\Psi}, \frac{\partial \mathbf{\Gamma}}{\partial n} \right) \right\} dS. \tag{3.15}$$

Let us prove the potentiality of the general representation (3.14) by performing differentiation with respect to the external variable, using the properties of the tensor Γ (1.31 - 1.33) and taking into account that:

$$\left[\nabla_y, \left(\left[\mathbf{n}, \left[\nabla, \mathbf{\Psi}\left(|\mathbf{x}| \right) \right] \right], \mathbf{\Gamma}\left(|\mathbf{x}-\mathbf{y}| \right) \right) \right] = \left(\left[\mathbf{n}, \left[\nabla, \mathbf{\Psi} \right] \right], \nabla \mathbf{G} \right) - \left[\left[\mathbf{n}, \left[\nabla, \mathbf{\Psi} \right] \right], \nabla \varphi \right]$$

and

$$\left[\nabla_y, \left(\mathbf{\Psi}\left(|\mathbf{x}| \right), \left[\mathbf{n}, \left[\nabla, \mathbf{\Gamma}\left(|\mathbf{x}-\mathbf{y}| \right) \right] \right] \right) \right] = \left(\mathbf{n}, \left[\mathbf{\Psi}, \nabla\nabla\varphi \right] \right).$$

Here

$$\left(\left[\mathbf{n}, \left[\nabla, \mathbf{\Psi} \right] \right], \nabla \mathbf{G} \right) = \left(\mathbf{n}, \left[\nabla, \left[\nabla, \mathbf{\Psi} \right] \right] \right) \mathbf{G} - \left(\mathbf{n}, \left[\nabla, \left(\left[\nabla, \mathbf{\Psi} \right] \mathbf{G} \right) \right] \right),$$

$$\left(\mathbf{n}, \left[\mathbf{\Psi}, \nabla\nabla\varphi \right] \right) = \left(\mathbf{n}, \left[\nabla, \mathbf{\Psi} \right] \right) \nabla\varphi - \left(\mathbf{n}, \left[\nabla, \left(\mathbf{\Psi} \nabla\varphi \right) \right] \right).$$

That is, the homogeneous integral equation obtained in this way can have only one zero solution, which proves the potentiality of the integral representation (3.14) and does not contradict the results (1.22, 1.23).

CONCLUSION

- A methodology for creating a mathematically correct method of boundary integral equations for solving boundary value problems of viscous gas dynamics in the real range of similarity criteria based on the developed and generalized apparatus of vector-tensor analysis and modern methods of mathematical physics is presented.
- Integral representations of solutions to spatial nonlinear boundary value problems of viscous gas dynamics are systematically studied within the framework of a mathematical model - the complete system of Navier-Stokes equations. It is substantiated that integral representations of solutions - as analytical expressions - may be of interest in the processes of designing vehicles and engineering structures with optimal aero- and hydrodynamic characteristics, as well as for the purpose of obtaining extreme values of such characteristics of load-bearing systems of aviation and rocket and space technology.
- For the first time, the reduction of the spatial boundary value problem of the dynamics of a viscous gas to a system of adequate boundary integral equations has been carried out; The differential properties of the kernels of generalized potentials of vector representations of velocity, vorticity, momentum vector potential, as well as the vector components of the momentum tensor were studied.
- It is noted that the results obtained convincingly confirm the fact of a constant historical supply of solutions to physical and mechanical problems for further

research/study of new classes of problems in mathematical physics (integral equations based on integral representations (3.5, 3.9, 3.13, 3.15)).As for the issue of the approximate solution of the system of linear boundary equations, the quadrature-interpolation method of calculating integrals on the triangulated boundary surfaces of the control volume, which leads to a uniquely solvable heterogeneous system of linear algebraic equations, can be considered the most reliable and tested [8].

- At the level of differential forms, it seems realistic to develop this methodology to solve current conservative initial-boundary value problems of non-stationary continuum mechanics.

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