The 9th European Congress on Computational Methods in Applied Sciences and Engineering
ECCOMAS Congress 2024
3–7 June 2024, Lisboa, Portugal

ON A SKIN EFFECT IN MAGNETIC CONDUCTORS

VICTOR PÉRON

Laboratoire de mathématiques appliquées de Pau, E2S UPPA, CNRS
Université de Pau et des Pays de l'Adour
64000 Pau, France
e-mail: victor.peron@univ-pau.fr

Key words: Maxwell equations, High relative permeability, Lipschitz domains, Asymptotic expansions, Impedance boundary conditions

Summary. We consider the time-harmonic Maxwell equations set on a domain which represents a magnetic conductor surrounded by a non-magnetic body such that the relative magnetic permeability μ_r between the materials is large. We state uniform a priori estimates for Maxwell transmission problem when the interface between the two subdomains is supposed to be Lipschitz. Assuming smoothness for the interface between the subdomains, we prove that the solution of the Maxwell equations admits a multiscale expansion in powers of $\varepsilon = 1/\sqrt{\mu_r}$ with profile terms rapidly decaying inside the magnetic conductor. As an application of uniform estimates, we develop an argument for the convergence of this expansion as ε tends to zero. We derive also impedance boundary conditions on the interface Σ, up to the third order of approximation with respect to ε.

1 INTRODUCTION

The computation of the electromagnetic field in strongly heterogeneous media is crucial in many subject areas in real-world applications, see, *e.g.*, [1, 2, 5] where additional references may also be found. This computation can be made particularly difficult due in particular to very high contrasts of magnetic permeabilities, see, *e.g.*, [5]. In this context, many works in the literature are related to issues of uniform estimates to tackle rigorously stiff transmission problems in electromagnetism with numerical methods. We refer the reader, for instance, to the works in Ref. [1, 3] for transmission problems with high contrast of electric conductivities.

In this work we investigate a transmission problem in materials presenting high contrast of magnetic permeabilities. We consider the time-harmonic Maxwell equations set on a domain made up of two subdomains that represent a magnetic conductor surrounded by a non-magnetic material, and such that the relative magnetic permeability μ_r between the materials can be very high.

We state uniform a priori estimates for the transmission problem when the interface between the two subdomains is supposed to be Lipschitz (Section 2). Then assuming smoothness for the interface between the subdomains, we prove that the solution of the Maxwell equations admits a multiscale expansion in powers of a small parameter $\varepsilon = 1/\sqrt{\mu_r}$, with *profiles* rapidly decaying inside the magnetic conductor (Section 3). This expansion allows to describe an anomalous *skin effect* which is different from the classical skin effect at high conductivity, see, *e.g.*, [1, 2]. The convergence of this expansion as ε tends to zero is based on uniform estimates. As a by-product of the multiscale expansion we infer also a new family of impedance boundary conditions

(IBCs) on the interface for the electromagnetic field (Section 4). IBCs are useful for instance for numerical purposes since they allow to reduce the computational domain, see, *e.g.*, [1]. We derive IBCs on the interface Σ up to the third order of approximation with respect to ε. The first order IBC is the perfectly insulating electric boundary condition

$$\mathbf{E}\cdot\mathbf{n}=0 \quad \text{and} \quad \mathbf{H}\times\mathbf{n}=0 \quad \text{on} \quad \Sigma\,.$$

The second order IBC writes

$$\mathbf{H}\times\mathbf{n}=\frac{1}{\sqrt{\mu_-}}\sqrt[4]{\varepsilon_0^2+\left(\frac{\sigma}{\omega}\right)^2}\,\mathrm{e}^{\frac{i}{2}\arctan\left(\frac{\sigma}{\omega\varepsilon_0}\right)}(\mathbf{n}\times\mathbf{E})\times\mathbf{n} \quad \text{on} \quad \Sigma\,. \tag{1}$$

The third order IBC writes

$$\mathbf{H}\times\mathbf{n}=\left(\frac{1}{\sqrt{\mu_-}}\sqrt[4]{\varepsilon_0^2+\left(\frac{\sigma}{\omega}\right)^2}\,\mathrm{e}^{\frac{i}{2}\arctan\left(\frac{\sigma}{\omega\varepsilon_0}\right)}+\frac{1}{i\omega\mu_-}(\mathcal{H}-\mathcal{C})\right)(\mathbf{n}\times\mathbf{E})\times\mathbf{n} \quad \text{on} \quad \Sigma\,. \tag{2}$$

Here $\mathcal{H}$ is the mean curvature of the surface Σ and $\mathcal{C}$ is a curvature tensor field defined on Σ.

2 FRAMEWORK AND UNIFORM ESTIMATES

Let Ω be a smooth bounded simply connected domain in $\mathbb{R}^3$, and $\Omega_-\subset\subset\Omega$ be a Lipschitz connected subdomain of Ω. We denote by Γ the boundary of Ω, and by Σ the boundary of Ω_-. Finally, we denote by Ω_+ the complementary of $\overline{\Omega}_-$ in Ω, cf. Figure 1.

We consider the time-harmonic Maxwell equations given by Faraday's and Ampère's laws in Ω:

$$\operatorname{curl}\ \mathbf{E}-i\omega\underline{\mu}\mathbf{H}=0 \quad \text{and} \quad \operatorname{curl}\ \mathbf{H}+(i\omega\varepsilon_0-\underline{\sigma})\mathbf{E}=\mathbf{j} \quad \text{in} \quad \Omega\,. \tag{3}$$

Here, $(\mathbf{E},\mathbf{H})$ represents the electromagnetic field, ε_0 is the electric permittivity, ω is the angular frequency ($\omega\neq 0$ and the time convention is $\exp(-i\omega t)$), $\mathbf{j}$ represents a current density and is supposed to belong to $\mathbf{L}^2(\Omega)$, $\underline{\sigma}$ is the electric conductivity, and $\underline{\mu}$ is the magnetic permeability. We assume that the coefficients $\underline{\sigma}$ and $\underline{\mu}$ take different values in Ω_+ and Ω_-, $(\sigma_+>0,\sigma_->0)$ and $(\mu_+>0,\mu_->0)$, respectively, and we denote by μ_r the relative magnetic permeability between the subdomains Ω_- and Ω_+, and which is defined as:

$$\mu_r=\mu_-/\mu_+\,. \tag{4}$$

We are particularly interested in the case where this parameter can be high, which is often the case when the domain Ω_- represents a linear magnetic conductive material and the domain Ω_+ represents a non-magnetic material.

To complement the Maxwell harmonic equations (3), we consider the perfectly insulating electric boundary conditions

$$\mathbf{E}\cdot\mathbf{n}=0 \quad \text{and} \quad \mathbf{H}\times\mathbf{n}=0 \quad \text{on} \quad \Gamma\,. \tag{5}$$

Hereafter, we denote by $\|\cdot\|_{0,\mathcal{O}}$ the norm in $\mathbf{L}^2(\mathcal{O})=\mathrm{L}^2(\mathcal{O})^3$. In the framework above it is possible to prove uniform a priori estimates for the Maxwell transmission problem, cf. [6] :

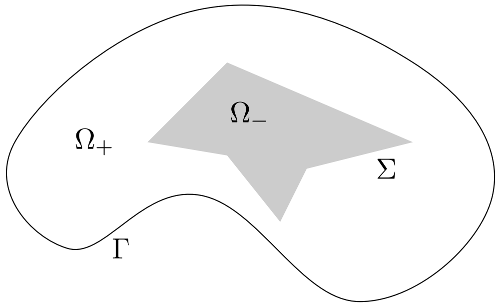

Figure 1: The domain Ω and its subdomains Ω_- and Ω_+

Theorem 2.1 *Let $\sigma_\pm > 0$, and $\mu_+ > 0$. If the interface Σ is Lipschitz, there are constants $\mu_\star$ and $C > 0$ such that for all $\mu_r \geqslant \mu_\star$ the Maxwell problem* (3) *with boundary conditions* (5) *and data $\mathbf{j} \in \mathbf{L}^2(\Omega)$ has a unique solution $(\mathbf{E}, \mathbf{H})$ in $\mathbf{L}^2(\Omega)^2$, which satisfies:*

$$\|\mathbf{H}\|_{0,\Omega} + \|\mathbf{E}\|_{0,\Omega} + \sqrt{\mu_r}\,\|\mathbf{H}\|_{0,\Omega_-} \leqslant C\|\mathbf{j}\|_{0,\Omega}. \tag{6}$$

The proof of this result requires more than six pages, it is worked out in detail in [6]. The proof is based on an appropriate decomposition of the magnetic field, whose gradient part is estimated thanks to uniform estimates for a scalar transmission problems with constant coefficients on two subdomains.

3 MULTISCALE EXPANSION

Several works are devoted to the question of an asymptotic expansion as $\mu_r \to \infty$ of solutions of eddy current problems *when the interface Σ is smooth*, see [3, 4, 5]. In this work we address the issue of an asymptotic expansion as $\mu_r \to \infty$ of solutions of the Maxwell system.

This section is concerned with a rigorous multiscale expansion for the magnetic field at high relative magnetic permeability. In Section 3.1 we provide the first terms of this expansion. Then in Section 3.2 we state error estimates for this expansion. We refer also the reader to the work in Ref. [6] where elements of proof for the multiscale expansion are given.

In that follows we assume that right-hand side $\mathbf{j}$ in (3) is smooth and that Assumption 3.1 holds:

Assumption 3.1 *We assume that the surfaces Σ (interface) and Γ (external boundary) are smooth.*

Let $\sigma_\pm > 0$, $\mu_+ > 0$. In that follows we work with the small parameter ε defined as

$$\varepsilon = \frac{1}{\sqrt{\mu_r}} > 0\ .$$

By Theorem 2.1 there exists $\varepsilon_\star > 0$ such that for all $\varepsilon \in (0, \varepsilon_\star)$, the Maxwell problem (3) with boundary conditions (5) has a unique solution $(\mathbf{E}_\varepsilon, \mathbf{H}_\varepsilon)$. In that follows, the magnetic field $\mathbf{H}_{(\varepsilon)}$ is denoted by $\mathbf{H}^+_{(\varepsilon)}$ in the non-magnetic part Ω_+, and by $\mathbf{H}^-_{(\varepsilon)}$ in the magnetic conducting part

Ω_-. Then both parts possess series expansions in powers of ε:

$$\mathbf{H}^+_{(\varepsilon)}(\mathbf{x}) \approx \sum_{j\geqslant 0} \varepsilon^j \mathbf{H}^+_j(\mathbf{x}) , \quad \mathbf{x} \in \Omega_+ , \tag{7}$$

$$\mathbf{H}^-_{(\varepsilon)}(\mathbf{x}) \approx \sum_{j\geqslant 0} \varepsilon^j \mathbf{H}^-_j(\mathbf{x};\varepsilon), \quad \mathbf{x} \in \Omega_- , \quad \text{with} \quad \mathbf{H}^-_j(\mathbf{x};\varepsilon) = \chi(y_3)\, \underline{\mathfrak{H}}_j(y_\alpha, \frac{y_3}{\varepsilon}) . \tag{8}$$

There hold a similar series expansions in powers of ε for the electric field. In (8), (y_α, y_3) is a *normal coordinate system* to the surface Σ which is defined in $\mathcal{U}_-$, a tubular neighborhood of the surface Σ in the domain Ω_- (cf. Figure 2): y_α $(\alpha = 1, 2)$ are tangential coordinates on Σ, and y_3 is the normal coordinate to Σ, cf. *e.g.*, [2, 6]. The function $\mathbf{y} \mapsto \chi(y_3)$ is a smooth cut-off with support in $\overline{\mathcal{U}}_-$ and equal to 1 in a smaller tubular neighborhood of Σ. The vector fields $\underline{\mathfrak{H}}_j : (y_\alpha, Y_3) \mapsto \underline{\mathfrak{H}}_j(y_\alpha, Y_3)$ are *profiles* defined on $\Sigma \times \mathbb{R}^+$: They are exponentially decreasing with respect to Y_3 and are smooth in all variables.

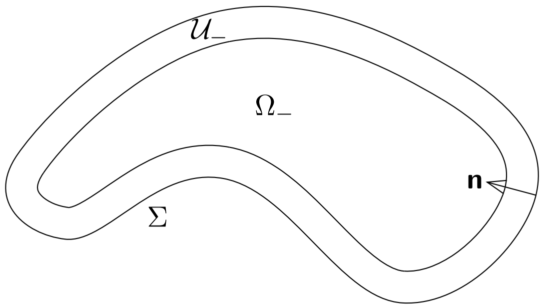

Figure 2: A tubular neighbourhood of the surface Σ

3.1 First terms of the multiscale expansion

In this section, we provide the construction of the first profiles $\underline{\mathfrak{H}}_j = (\mathfrak{H}_j, \mathfrak{h}_j)$ and of the first asymptotics $\mathbf{H}^+_j$. The first terms of this expansion are derived explicitly in [6].

The first profile $\underline{\mathfrak{H}}_0$ in the magnetic conductor is zero:

$$\underline{\mathfrak{H}}_0 = 0 . \tag{9}$$

Then, the first term of the magnetic field in the non-magnetic region solves Maxwell equations with perfectly insulating electric boundary conditions on $\partial\Omega_+ = \Sigma \cup \Gamma$:

$$\begin{cases} \operatorname{curl}\operatorname{curl}\mathbf{H}^+_0 - \kappa_+^2(1 + \frac{i}{\delta_+^2})\mathbf{H}^+_0 = \operatorname{curl}\mathbf{j} & \text{in} \quad \Omega_+ \\ \mathbf{H}^+_0 \times \mathbf{n} = 0 & \text{on} \quad \Sigma \cup \Gamma. \end{cases}$$

Here κ_+ is a real wavenumber defined by $\kappa_+ = \omega\sqrt{\varepsilon_0\mu_+}$.

The first profile in the magnetic region is a tangential field which is exponential with a complex rate λ:

$$\mathfrak{H}_1(y_\alpha, Y_3) = -\mathrm{j}_0(y_\alpha)\, \mathrm{e}^{-\lambda Y_3} . \tag{10}$$

Here $\mathrm{j}_0(y_\alpha) = \lambda^{-1}(1+\frac{i}{\delta_-^2})(1+\frac{i}{\delta_+^2})^{-1}\left(\operatorname{curl}\mathbf{H}_0^+\times\mathbf{n}\right)(y_\alpha,0)$, and λ is given by:

$$\lambda = \kappa_+\sqrt[4]{1+\frac{1}{\delta_-^4}}\,\mathrm{e}^{i\frac{\theta(\delta_-)-\pi}{2}} \quad \text{with} \quad \delta_- = \sqrt{\omega\varepsilon_0/\sigma_-}\ , \quad \text{and} \quad \theta(\delta_-) = \arctan\frac{1}{\delta_-^2}\,. \tag{11}$$

Note that $\operatorname{Re}\lambda > 0$, and if j_0 is not identically 0, there exists a constant $C > 0$ independent of ε such that

$$C^{-1}\sqrt{\varepsilon} \leqslant \|\mathbf{H}_1^-(\,\cdot\,;\varepsilon)\|_{0,\Omega_-} \leqslant C\sqrt{\varepsilon}\ . \tag{12}$$

Note also that the normal component of the profile $\underline{\mathfrak{H}}_1$ is zero:

$$\mathfrak{h}_1 = 0\,.$$

The next term in the non-magnetic region solves:

$$\begin{cases} \operatorname{curl}\operatorname{curl}\mathbf{H}_1^+ - \kappa_+^2(1+\frac{i}{\delta_+^2})\mathbf{H}_1^+ = 0 & \text{in} \quad \Omega_+ \\ \mathbf{H}_1^+\times\mathbf{n} = -\mathrm{j}_0\times\mathbf{n} & \text{on} \quad \Sigma \\ \mathbf{H}_1^+\times\mathbf{n} = 0 & \text{on} \quad \Gamma. \end{cases}$$

Like above, define j_1 as $\mathrm{j}_1(y_\alpha) = \lambda^{-1}(1+\frac{i}{\delta_-^2})(1+\frac{i}{\delta_+^2})^{-1}\left(\operatorname{curl}\mathbf{H}_1^+\times\mathbf{n}\right)(y_\alpha,0)$ on the interface Σ. Then, the tangential components of the profile $\underline{\mathfrak{H}}_2$ are given by the tangential field $\mathfrak{H}_2$:

$$\mathfrak{H}_2(y_\alpha,Y_3) = \Big[-\mathrm{j}_1 + \left(\lambda^{-1}+Y_3\right)(\mathcal{C}-\mathcal{H})\,\mathrm{j}_0\Big](y_\alpha)\ \mathrm{e}^{-\lambda Y_3}\,. \tag{13}$$

Here $\mathcal{H} = \frac{1}{2}\,b_\alpha^\alpha$ is the *mean curvature* of the surface Σ[1], and $\mathcal{C}$ is the curvature tensor field on Σ defined by

$$(\mathcal{C}\mathrm{j})_\alpha = b_\alpha^\beta\,\mathrm{j}_\beta\,, \tag{14}$$

with $b_\alpha^\beta = a^{\beta\gamma}b_{\gamma\alpha}$, and $a^{\beta\gamma}$ is the inverse of the metric tensor $a_{\beta\gamma}$ in Σ, and $b_{\gamma\alpha}$ is the curvature tensor in Σ. The next term which is determined is the normal component $\mathfrak{h}_2$ of the profile $\underline{\mathfrak{H}}_2$:

$$\mathfrak{h}_2(y_\alpha,Y_3) = -\lambda^{-1}\operatorname{div}_\Sigma\ \mathrm{j}_0(y_\alpha)\,\mathrm{e}^{-\lambda Y_3}\,.$$

Here $\operatorname{div}_\Sigma$ is the surface divergence operator on Σ. The next term in the non-magnetic region solves :

$$\begin{cases} \operatorname{curl}\operatorname{curl}\mathbf{H}_2^+ - \kappa_+^2(1+\frac{i}{\delta_+^2})\mathbf{H}_2^+ = 0 & \text{in} \quad \Omega_+ \\ \mathbf{n}\times\mathbf{H}_2^+\times\mathbf{n} = -\mathrm{j}_1 + \lambda^{-1}(\mathcal{C}-\mathcal{H})\,\mathrm{j}_0 & \text{on} \quad \Sigma \\ \mathbf{H}_2^+\times\mathbf{n} = 0 & \text{on} \quad \Gamma. \end{cases}$$

3.2 Validation of the multiscale expansion

The validation of the multiscale expansion (7)-(8) for the magnetic field $\mathbf{H}_{(\varepsilon)}$ consist in proving estimates for remainders $\mathbf{R}_{m;\,\varepsilon}$ which are defined as

$$\mathbf{R}_{m;\,\varepsilon} = \mathbf{H}_{(\varepsilon)} - \sum_{j=0}^{m}\varepsilon^j\mathbf{H}_j \quad \text{in} \quad \Omega\,. \tag{15}$$

We quote from [6] the following result.

[1] In particular, the sign of $\mathcal{H}$ depends on the orientation of the surface Σ. As a convention, the unit normal vector $\mathbf{n}$ on the surface Σ is inwardly oriented to Ω_-, see Figure 2.

Theorem 3.2 *In the framework above, for all $m \in \mathbb{N}$ and $\varepsilon \in (0, \varepsilon_0]$, the remainder $\mathbf{R}_{m;\,\varepsilon}$ (15) satisfies the optimal estimate*

$$\|\mathbf{R}^+_{m;\,\varepsilon}\|_{0,\Omega_+} + \|\operatorname{curl}\mathbf{R}^+_{m;\,\varepsilon}\|_{0,\Omega_+} + \varepsilon^{-\frac{1}{2}}\|\mathbf{R}^-_{m;\,\varepsilon}\|_{0,\Omega_-} + \varepsilon^{\frac{1}{2}}\|\operatorname{curl}\mathbf{R}^-_{m;\,\varepsilon}\|_{0,\Omega_-} \leqslant C_m\varepsilon^{m+1}. \quad (16)$$

This result is proved in [6]. The proof is based on an evaluation of the right hand side when the Maxwell operator is applied to $\mathbf{R}_{m;\,\varepsilon}$ and on uniform estimates for $(\mathbf{E}_\varepsilon, \mathbf{H}_\varepsilon)$ (cf. Thm. 2.1, Section 2). By construction of the multiscale expansion, we obtain (cf. [6])

$$\begin{cases} \operatorname{curl}\alpha_+^{-1}\operatorname{curl}\mathbf{R}^+_{m;\,\varepsilon} - \kappa_+^2\mathbf{R}^+_{m;\,\varepsilon} & = & 0 & \text{in} & \Omega_+ \\ \operatorname{curl}\alpha_-^{-1}\operatorname{curl}\mathbf{R}^-_{m;\,\varepsilon} - \varepsilon^{-2}\kappa_+^2\mathbf{R}^-_{m;\,\varepsilon} & = & \mathbf{j}^-_{m;\,\varepsilon} & \text{in} & \Omega_- \\ \big[\mathbf{R}_{m;\,\varepsilon}\times\mathbf{n}\big]_\Sigma & = & 0 & \text{on} & \Sigma \\ \big[\underline{\alpha}^{-1}\operatorname{curl}\mathbf{R}_{m;\,\varepsilon}\times\mathbf{n}\big]_\Sigma & = & \mathbf{g}_{m;\,\varepsilon} & \text{on} & \Sigma \\ \mathbf{R}^+_{m;\,\varepsilon}\times\mathbf{n} & = & 0 & \text{on} & \partial\Omega\,. \end{cases} \quad (17)$$

Here, $\underline{\alpha} = (\alpha_+, \alpha_-)$, $\alpha_+ = 1 + i/\delta_+^2$ and $\alpha_- = 1 + i/\delta_-^2$, and $[\mathbf{H}\times\mathbf{n}]_\Sigma$ denotes the jump of $\mathbf{H}\times\mathbf{n}$ across Σ. The right hand sides (residues) $\mathbf{j}^-_{m;\,\varepsilon}$ and $\mathbf{g}_{m;\,\varepsilon}$ are, roughly, of the order ε^m : we have $\mathbf{j}^-_{0;\,\varepsilon} = 0$ and $\mathbf{g}_{0;\,\varepsilon} = \mathcal{O}(1)$; for all $m \in \mathbb{N}\setminus\{0\}$, we have $\mathbf{j}^-_{m;\,\varepsilon} = \mathcal{O}(\varepsilon^{m-1})$ and $\mathbf{g}_{m;\,\varepsilon} = \mathcal{O}(\varepsilon^m)$, and we have the following estimates

$$\|\mathbf{j}^-_{m;\,\varepsilon}\|_{0,\Omega_-} \leqslant C_m\varepsilon^{m-1} \quad \text{and} \quad \|\mathbf{g}_{m;\,\varepsilon}\|_{\frac{1}{2},\Sigma} \leqslant C_m\varepsilon^m, \quad (18)$$

where $C_m > 0$ is independent of ε.

4 IMPEDANCE BOUNDARY CONDITIONS

As a by-product of the asymptotic expansions we infer a new family of impedance boundary conditions (IBCs) for the electromagnetic field. In this section we summarize the method for deriving the family of IBCs and we state the IBCs up to the third order of approximation. The construction and the validation of the IBCs as well as numerical experiments to assess their accuracy will be detailed in a forthcoming paper.

4.1 Derivation of impedance boundary conditions

The proof is based on the derivation of a simpler problem which is satisfied by the truncated expansions

$$\mathbf{E}^+_{k,\varepsilon} := \mathbf{E}^+_0 + \varepsilon\mathbf{E}^+_1 + \varepsilon^2\mathbf{E}^+_2 + \cdots + \varepsilon^k\mathbf{E}^+_k \quad \text{in} \quad \Omega_+$$

and

$$\mathbf{H}^+_{k,\varepsilon} := \mathbf{H}^+_0 + \varepsilon\mathbf{H}^+_1 + \varepsilon^2\mathbf{H}^+_2 + \cdots + \varepsilon^k\mathbf{H}^+_k \quad \text{in} \quad \Omega_+\ ,$$

up to a residual term in $\mathcal{O}(\varepsilon^{k+1})$, for $k = 0, 1, 2$.

By construction of the asymptotic expansions, we obtain

$$\begin{cases} \operatorname{curl}\mathbf{E}_{k,\varepsilon} - i\omega\mu_+\mathbf{H}_{k,\varepsilon} & = & 0 & \text{in} & \Omega_+ \\ \operatorname{curl}\mathbf{H}_{k,\varepsilon} + (i\omega\varepsilon_0 - \sigma_+)\mathbf{E}_{k,\varepsilon} & = & \mathbf{j} & \text{in} & \Omega_+ \\ \mathbf{H}_{k,\varepsilon}\times\mathbf{n} & = & \mathsf{D}_{k,\varepsilon}\left((\mathbf{n}\times\mathbf{E}_{k,\varepsilon})\times\mathbf{n}\right) + \mathcal{O}(\varepsilon^{k+1}) & \text{on} & \Sigma \\ \mathbf{E}_{k,\varepsilon}\cdot\mathbf{n} & = & 0 & \text{on} & \partial\Omega \\ \mathbf{H}_{k,\varepsilon}\times\mathbf{n} & = & 0 & \text{on} & \partial\Omega\ , \end{cases} \quad (19)$$

with $\mathsf{D}_{k,\varepsilon}$ a surface differential operator.

Then by neglecting the residual term in $\mathcal{O}(\varepsilon^{k+1})$ in (19) we identify the simpler problem

$$\begin{cases} \operatorname{curl} \mathbf{E}_k^\varepsilon - i\omega\mu_+ \mathbf{H}_k^\varepsilon & = & 0 & \text{in} & \Omega_+ \\ \operatorname{curl} \mathbf{H}_k^\varepsilon + (i\omega\varepsilon_0 - \sigma_+)\mathbf{E}_k^\varepsilon & = & \mathbf{j} & \text{in} & \Omega_+ \\ \mathbf{H}_k^\varepsilon \times \mathbf{n} & = & \mathsf{D}_{k,\varepsilon}\left((\mathbf{n}\times\mathbf{E}_k^\varepsilon)\times\mathbf{n}\right) & \text{on} & \Sigma \\ \mathbf{E}_k^\varepsilon \cdot \mathbf{n} & = & 0 & \text{on} & \partial\Omega \\ \mathbf{H}_k^\varepsilon \times \mathbf{n} & = & 0 & \text{on} & \partial\Omega\ . \end{cases} \tag{20}$$

4.2 Statement of impedance boundary conditions

We obtain a hierarchy of IBCs up to the third order of approximation in ε.

First order

The first order IBC (i.e. for $k = 0$ in (20)) is the perfectly insulating electric boundary condition

$$\mathbf{H}_0 \times \mathbf{n} = 0 \quad \text{and} \quad \mathbf{E}_0 \cdot \mathbf{n} = 0 \quad \text{on} \quad \Sigma\ .$$

Second order

The second order IBC (i.e. for $k = 1$ in (20))) writes

$$\mathbf{H}_1^\varepsilon \times \mathbf{n} = -\frac{\varepsilon\lambda}{i\omega\mu_+}(\mathbf{n}\times\mathbf{E}_1^\varepsilon)\times\mathbf{n} \quad \text{on} \quad \Sigma\ .$$

Third order

The third order IBC (i.e. for $k = 2$ in (20))) writes

$$\mathbf{H}_2^\varepsilon \times \mathbf{n} = -\frac{\varepsilon\lambda}{i\omega\mu_+}\Big(1 - \frac{\varepsilon}{\lambda}\left(\mathcal{H} - \mathcal{C}\right)\Big)(\mathbf{n}\times\mathbf{E}_2^\varepsilon)\times\mathbf{n} \quad \text{on} \quad \Sigma\ .$$

Finally coming back to the definitions of λ and ε (cf. Section 3), we identify the IBCs (1)-(2) that are given in the introduction.

4.3 Validation of impedance boundary conditions

The validation of theses IBCs consist in proving uniform stability and convergence results for the solution of the simpler problem (20). We state without proof the following result.

Theorem 4.1 *In the framework of Section 3, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, for all $k \in \{0, 1, 2\}$, the problem (20) with data $\mathbf{j} \in \mathbf{L}^2(\Omega_+)$ has a unique solution $(\mathbf{E}_k^\varepsilon, \mathbf{H}_k^\varepsilon)$ in $(\mathbf{L}^2(\Omega_+))^2$, and*

$$\|\mathbf{E}_k^\varepsilon\|_{0,\Omega_+} + \|\mathbf{H}_k^\varepsilon\|_{0,\Omega_+} \leqslant C\|\mathbf{j}\|_{0,\Omega_+} \tag{21}$$

$$\|\mathbf{E}_\varepsilon - \mathbf{E}_k^\varepsilon\|_{0,\Omega_+} + \|\mathbf{H}_\varepsilon - \mathbf{H}_k^\varepsilon\|_{0,\Omega_+} \leqslant C_k\varepsilon^{k+1}\ . \tag{22}$$

The proof of this result will be detailed in a forthcoming paper. The convergence result (22) can be obtained as a consequence of the stability result (21) by using the truncated expansions $(\mathbf{E}^+_{k,\varepsilon}, \mathbf{H}^+_{k,\varepsilon})$ as intermediate quantities, together with estimates for remainders (cf. Thm. 3.2, Section 3).

5 CONCLUSIONS

In this work we tackled an interface problem in materials with high contrast in magnetic permeabilities. We proved uniform a priori estimates for the electromagnetic field solution of the time-harmonic Maxwell equations as the relative magnetic permeability μ_r between a magnetic conductor and a non-magnetic material tends to infinity, and when the interface between the subdomains is Lipschitz. Assuming smoothness for the interface between the subdomains, we described a skin effect with a rigorous multiscale expansion for the magnetic field in powers of $\frac{1}{\sqrt{\mu_r}}$ with profiles inside the magnetic conductor. We derived also a new family of impedance boundary conditions up to the third order of approximation with respect to $\frac{1}{\sqrt{\mu_r}}$ for the electromagnetic field.

REFERENCES

[1] Haddar, H., Joly, P. and Nguyen, H.-M. (2008). Generalized impedance boundary conditions for scattering problems from strongly absorbing obstacles: the case of Maxwell's equations. *Math. Models Methods Appl. Sci.*, **18**(10):1787–1827

[2] Caloz, G., Dauge, M., Faou, E. and Péron, V. (2011). On the influence of the geometry on skin effect in electromagnetism. *Computer Methods in Applied Mechanics and Engineering*, **200**(9-12):1053–1068

[3] Péron, V. and Poignard, C. (2021). On a magnetic skin effect in eddy current problems: the magnetic potential in magnetically soft materials. *Zeitschrift für angewandte Mathematik und Physik*, **72**(4), 164

[4] Abou El Nasser El Yafi, D., Péron, V., Perrussel, R. and Krähenbühl, L. (2023). Numerical study of the magnetic skin effect: Efficient parameterization of 2D surface-impedance solutions for linear ferromagnetic materials. *International Journal of Numerical Modelling: Electronic Networks, Devices and Fields*, **36**(3), e3051

[5] Abou El Nasser El Yafi, D. and Péron, V. (2023). Efficient asymptotic models for axisymmetric eddy current problems in linear ferromagnetic materials. *Computers & Mathematics with Applications*, **151**: 335–345

[6] Péron, V. (2024). Uniform estimates for transmission problems in electromagnetism with high contrast in magnetic permeabilities, arXiv preprint arXiv:2406.15276