16th World Congress in Computational Mechanics (WCCM)
21-26 July 2024, Vancouver, Canada
A. Korobenko, M. Laforest, S. Prudhomme, R. Vaziri (Eds)

SPACE-TIME ANALYSIS FOR THE CONTAINER PROBLEM

M. GÖTTLICHER

University of Applied Sciences Erfurt
Altonaer Str. 25, 99094 Erfurt, Gemany
m.goettlicher@t-online.de, www.fh-erfurt.de

Key Words: Potential, Darcy, Worktime.

Abstract. The container problem describes the behaviour of elastic porous media in a rectangular container, which is completely saturated by an ideal incompressible liquid. By time the liquid extrudes on the surface of the container while the stress resulting from a given top load acts on the shrinking elastic solid due to its compression. The analysis bases on a space-time potential that contains a linear elastic term, a darcy flow term, and a boundary load term. The variation of the potential results in a space-time principle. Its minimum preserves approximately equilibrium over space and time.

1 INTRODUCTION

The container problem describes the behaviour of elastic porous media in a rectangular container, which is completely saturated by an ideal incompressible liquid. Displacement and velocity at the bottom are zero. Motion is possible only in the vertical direction. The boundary load on the top acts on the elastic solid. It starts with zero at the beginning and increases to its maximum at the end of duration T. By time the liquid extrudes on the surface of the container while the stress moves to the shrinking elastic solid due to its compression [2], [4].

Quadratic approximation of the displacement in space results in linear elastic stress distribution and cubic liquid pressure distribution. It fulfils compatibility of displacement, strain, and velocity. The cubic approximation of the displacement in time starts with a time gradient of zero and ends towards infinity with a time gradient of zero. Displacements and velocities are zero in the beginning.

The state at the end of duration T results from the minimum of virtual worktime which is the integral of virtual work over space and time. For T towards infinity solutions are available for low order approximation according to the rule of Bernoulli L' Hospital. The potential decreases for higher order approximation which indicates convergence [1], [5].

2 POTENTIAL AND PRINCIPLE

2.1 Space Formulation

For a given point of time the elastic stripe is subjected to a downward (negative) top load and an upward area load which results from the declining liquid pressure resulting from the velocity of the upward moving water.

Potential:

$$\frac{1}{2}\int_A \boldsymbol{\varepsilon}^T \boldsymbol{\sigma}\, dA - \frac{1}{2}\int_A \mathbf{u}\,\mathbf{p}'\, dA - \int_B \mathbf{u}\,\boldsymbol{\sigma}\, dB \tag{1}$$

$$\boldsymbol{\varepsilon} = \mathbf{D}\mathbf{u} = \begin{pmatrix} \dfrac{\partial u_x}{\partial x} \\ \dfrac{\partial u_y}{\partial y} \\ \dfrac{\partial u_x}{\partial y} + \dfrac{\partial u_y}{\partial x} \end{pmatrix} \qquad \boldsymbol{\sigma} = \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{pmatrix} \qquad \mathbf{u} = \begin{pmatrix} u_x \\ u_y \end{pmatrix} \qquad \mathbf{p}' = \begin{pmatrix} p'_x \\ p'_y \end{pmatrix} \tag{2}$$

$\boldsymbol{\varepsilon}$ strain

$\boldsymbol{\sigma}$ stress

$\mathbf{u}$ displacement

$\mathbf{p}'$ pressure gradient

A area

B boundary

The potential contains an elastic term, an area load term, and a boundary load term [3]. The area load results from the impact of the declining pressure on the elastic solid. The boundary load is given and identical to the elastic stress of the top. The liquid pressure is zero at the top.

Principle:

$$\int_A \delta\boldsymbol{\varepsilon}^T \boldsymbol{\sigma}\, dA - \frac{1}{2}\int_A \delta\mathbf{u}\,\mathbf{p}'\, dA - \frac{1}{2}\int_A \delta\mathbf{p}'\,\mathbf{u}\, dA - \int_B \delta\mathbf{u}\,\boldsymbol{\sigma}\, dB = 0 \tag{3}$$

$$\boldsymbol{\sigma} = \mathbf{E}\,\boldsymbol{\varepsilon} \tag{4}$$

$$\mathbf{p}' = -\mathbf{K}\,\dot{\mathbf{u}} \tag{5}$$

$$\mathbf{E} = \frac{E}{(1+\nu)(1-2\nu)} \cdot \begin{pmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \dfrac{1-2\nu}{2} \end{pmatrix} \qquad \mathbf{K} = \frac{1}{k} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{6}$$

$$\int_A \delta(\mathbf{D}\mathbf{u})^T \mathbf{E}(\mathbf{D}\mathbf{u})\, dA + \frac{1}{2}\int_A \delta\mathbf{u}\,\mathbf{K}\,\dot{\mathbf{u}}\, dA + \frac{1}{2}\int_A \delta\dot{\mathbf{u}}\,\mathbf{K}\mathbf{u}\, dA - \int_B \delta\mathbf{u}\,\boldsymbol{\sigma}\, dB = 0 \tag{7}$$

E Young's modulus

ν Poisson ratio

k Flow parameter

The pressure gradient depends on the displacement. For asymmetric interpolation in space and time evolve two area load terms.

Plane strain stripe of hight H and ν = 0:

$$E\int_0^H \delta u' u' \, dy + \frac{1}{2k}\int_0^H \delta \dot{u} u \, dy + \frac{1}{2k}\int_0^H \delta u \dot{u} \, dy - \delta u_H \sigma_H = 0 \tag{8}$$

$$u' = \frac{\partial u}{\partial y} \qquad \dot{u} = \frac{\partial u}{\partial t} \tag{9}$$

σ_H Stress at the top

u_H displacement at the top

H height

2.2 Space Time Formulation

The integration over space is extended by an integration over time.
Potential:

$$\frac{1}{2}\int_T\int_A \boldsymbol{\varepsilon}^T \boldsymbol{\sigma} \, dA \, dT - \frac{1}{2}\int_T\int_A \mathbf{u}\,\mathbf{p}' \, dA \, dT - \int_T\int_B \mathbf{u}\,\boldsymbol{\sigma} \, dB \, dT = 0 \tag{10}$$

T duration

Elastic stripe:

$$\frac{E}{2}\int_0^T\int_0^H u' u' \, dy \, dt + \frac{1}{2k}\int_0^T\int_0^H \dot{u} u \, dy \, dt - \int_0^T u_H \sigma_H \, dt \tag{11}$$

The analysis bases on the minimum of the space time potential.

Principle:

For a given duration T the virtual work is integrated over space and time The virtual work time is set to zero:

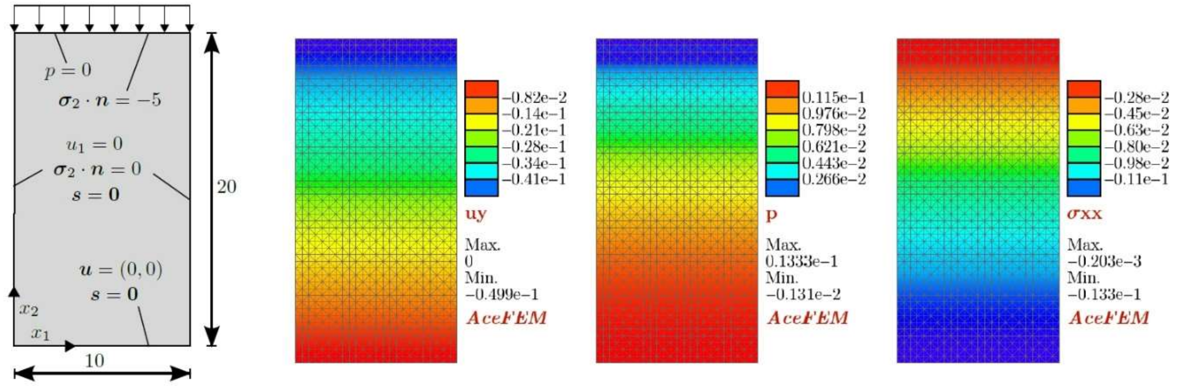
$$\int_T\int_A \delta\boldsymbol{\varepsilon}^T \boldsymbol{\sigma} \, dA \, dT - \int_T\int_A \delta\mathbf{u}\,\mathbf{p}' \, dA \, dT - \int_T\int_A \delta\mathbf{p}'\,\mathbf{u} \, dA \, dT - \int_T\int_B \delta\mathbf{u}\,\boldsymbol{\sigma} \, dB \, dT = 0 \tag{12}$$

Elastic stripe:

$$E\int_0^T\int_0^H \delta u' u' \, dy \, dt + \frac{1}{2k}\int_0^T\int_0^H \delta \dot{u} u \, dy \, dt + \frac{1}{2k}\int_0^T\int_0^H \delta u \dot{u} \, dy \, dt - \int_0^T \delta u_H \sigma_H \, dt = 0 \tag{13}$$

3 APPLICATIONS

A plane container of a height H of 20 meters and width L of 10 meters is considered.



Geometry and boundary conditions (left), displacement u_y, pressure p and stress σ_{xx} obtained with LSFEM for $k_L = 10^{-6}$ m⁴/Ns at time $T_{max} = 5 \times 10^5$ s (right)

Figure 1: Benchmark system [4].

$$\sigma_2 = -5\left[\frac{N}{m^2}\right] \qquad E = 2000\left[\frac{N}{m^2}\right] \qquad \nu = 0 \qquad k = 10^{-6}\left[\frac{m^4}{N\,s}\right] \qquad H = 20[m] \tag{14}$$

A constant top load acts on solid and liquid. In the beginning it acts completely at the liquid. By time the liquid extrudes on the surface of the container while the stress moves to the shrinking elastic solid due to its compression [4].

With respect to the presented time approximation in this paper the system is modified. Instead of a constant top load σ from the beginning to infinity, a time consistent load increasing from zero to maximum during the time T is applied. The flow parameter is reduced to 10^{-5}. Also, no top load acts on the liquid since the presented potential is restricted to the elastic solid. The liquid pressure p at the top is zero and it increases according to the upward velocity of the liquid towards the bottom. It is a secondary value.

3.1 Bilinear approximation

The rough bilinear approximation is appropriate to explore the solution mechanism of the space time analysis. Despite the significant violation of boundary conditions and equilibrium the main behavior is covered sufficiently. The power of the potential reduces the incompetence of the interpolation. The handmade analysis is easy to perform and to understand.

Displacement:

$$u = r \cdot u_H \tag{15}$$

$$u_H = s \cdot u_{HT} \qquad u_T = r \cdot u_{HT} \tag{16}$$

$$s = \frac{t}{T} \qquad r = \frac{y}{H} \tag{17}$$

$$0 \le s \le T \qquad 0 \le r \le H \tag{18}$$

$$u = r \cdot s \cdot u_{HT} \tag{19}$$

u_T displacement at the end

u_{HT} displacement at the top at the end

Strain:

$$u' = \frac{1}{H} \cdot s \cdot u_{HT} \qquad u'_{H0} = 0 \qquad u'_{HT} = \frac{1}{H} \cdot u_{HT} \qquad u'_{0T} = \frac{1}{H} \cdot u_{HT} \qquad u'_{00} = 0 \tag{20}$$

Velocity:

$$\dot{u} = \frac{1}{T} \cdot r \cdot u_{HT} \qquad \dot{u}_{H0} = 0 \qquad \dot{u}_{HT} = \frac{1}{T} \cdot u_{HT} \qquad \dot{u}_{0T} = \frac{1}{T} \cdot u_{HT} \qquad \dot{u}_{00} = 0 \tag{21}$$

u_{H0} displacement at the top at the beginning

u_{0T} displacement at the bottom at the end

u_{00} displacement at the bottom at the beginning

Stress:

$$\sigma_H = s \cdot \sigma_{HT} \tag{22}$$

σ_{HT} stress at the top at the end

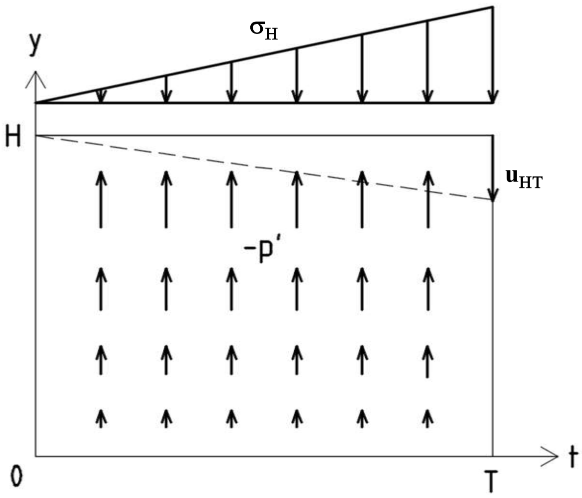


Figure 2: Space-time container problem.

Velocity and pressure gradient are linear over space and constant over time.

$$\begin{aligned} &EH\int_0^1\int_0^1 \frac{s^2}{H^2} d r T d s u_{HT} + \frac{H}{2k}\int_0^1\int_0^1 \frac{r}{T} r s d r T d s u_{HT} + \ldots \\ &\ldots + \frac{H}{2k}\int_0^1\int_0^1 r s \frac{r}{T} d r T d s u_{HT} - \int_0^1 s \cdot s \cdot \sigma_{HT} T d s = 0 \end{aligned} \tag{23}$$

$$E\,H\int_0^1 \frac{s^2}{H^2}\,T\,ds\,u_{HT} + \frac{H}{k}\int_0^1 \frac{s}{3}\,ds\,u_{HT} - \int_0^1 s\cdot s\cdot \sigma_{HT}\,T\,ds = 0 \tag{24}$$

$$\left(\frac{E\,T}{3\,H} + \frac{H}{6\,k}\right)\cdot u_{HT} = \frac{\sigma_{HT}\,T}{3} \tag{25}$$

$$u_{HT} = \frac{2\,H\,T\,k\,\sigma_{HT}}{2\,E\,T\,k + H^2} \tag{26}$$

$$u_{HT} = -0{,}04545 \tag{27}$$

Secondary analysis:

$$p = H\int_0^r p'\,dr - H\int_0^1 p'\,dr = \frac{H}{k}\int_0^r \dot{u}\,dr - \frac{H}{k}\int_0^1 \dot{u}\,dr = \frac{H\left(r^2-1\right)}{2\,T\,k}\cdot u_{HT} \tag{28}$$

Benchmark system:

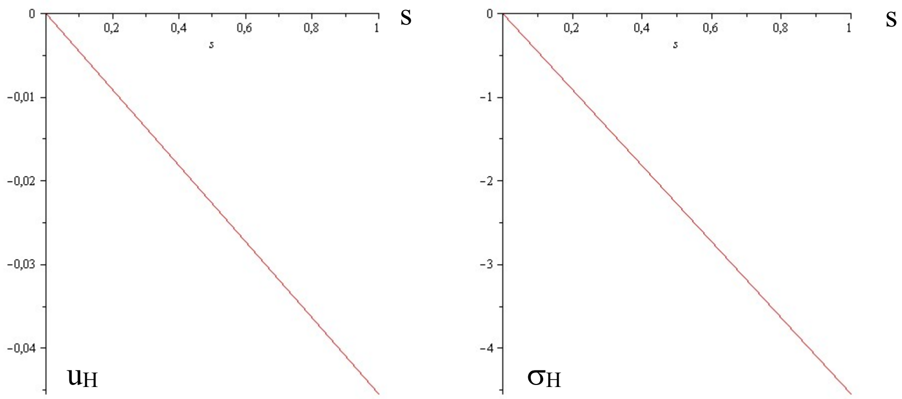


Figure 3: Displacement u_H and stress σ_H at the top.

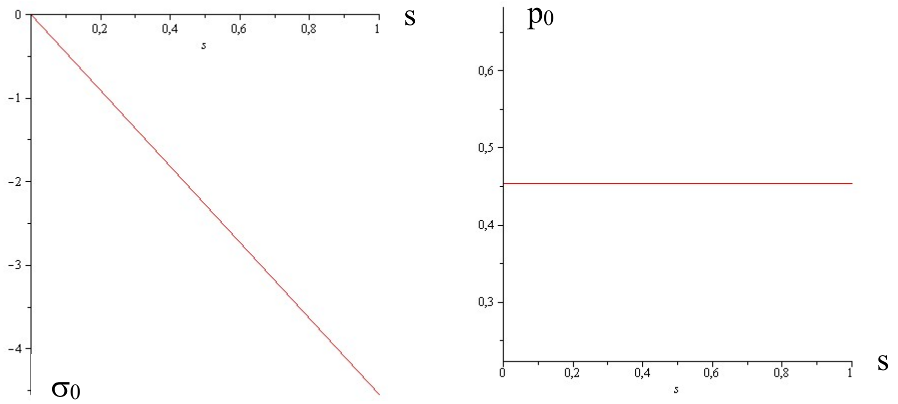


Figure 4: Stress σ_0 and pressure p_0 the bottom.

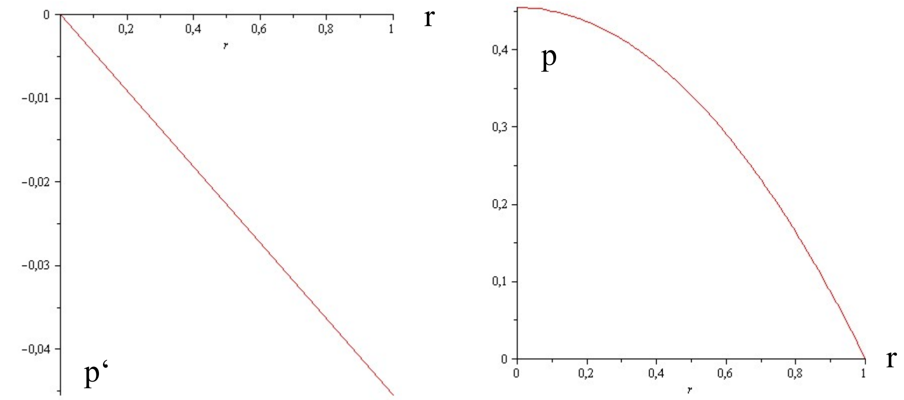


Figure 5: Pressure gradient p' and pressure p (constant in time).

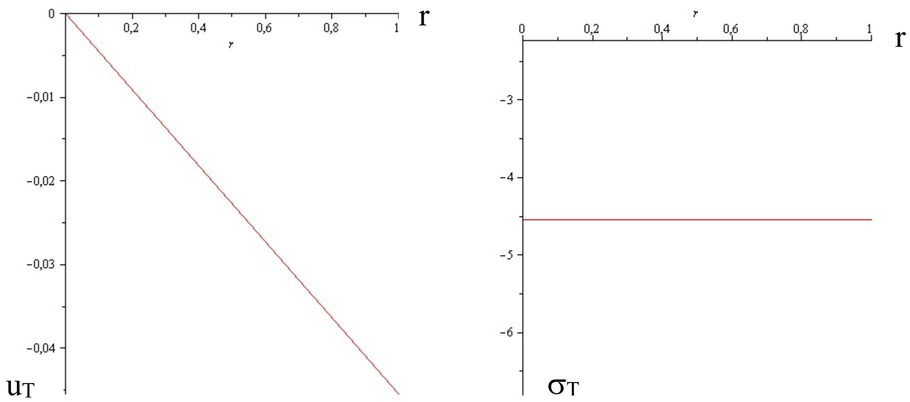


Figure 6: Displacement u_T and stress σ_T at the end of duration T.

$T \to \infty$

$$u_{H\infty} = \frac{H\,\sigma_{HT}}{E} \tag{29}$$

$$u_{H\infty} = -0,05 \tag{30}$$

Towards infinity the mere elastic solution is realized.

Mere Darcy solution:

$$\frac{H}{6k} \cdot u_{HT} - \frac{\sigma_{HT} T}{3} = 0 \tag{31}$$

$$u_{HT} = \frac{2k\,T\,\sigma_{HT}}{H} = -\frac{2 \cdot 0,00001 \cdot 100000 \cdot 5}{20} = 0,5 \tag{32}$$

Duration to reach the mere elastic displacement:

$$T = \frac{H\,u_{HT}}{2k\,\sigma_{HT}} = -\frac{20 \cdot 0,05}{2 \cdot 0,00001 \cdot 5} = 10000 \tag{33}$$

3.2 Quadratic cubic approximation

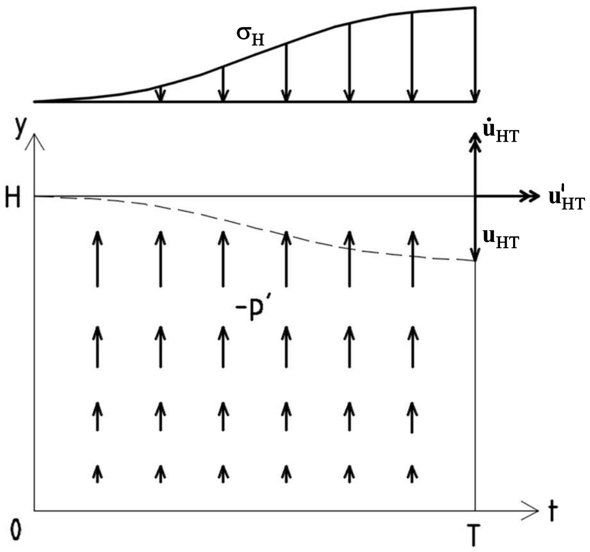


Figure 7: Quadratic cubic approximation.

$$u = \mathbf{f}^T\mathbf{u} = \begin{pmatrix} (2-r)r\cdot(3s^2-2s^3) \\ (-1+r)Hr\cdot(3s^2-2s^3) \\ (2-r)\cdot r\cdot(-s^2+s^3)\cdot T \end{pmatrix} \cdot \begin{pmatrix} u_{HT} \\ u'_{HT} \\ \dot{u}_{HT} \end{pmatrix} \tag{34}$$

Strain and velocity:

$$u' = \mathbf{f}'^T\mathbf{u} = \begin{pmatrix} (2-2r)\cdot(3s^2-2s^3) \\ (-1+2r)H\cdot(3s^2-2s^3) \\ (2-2r)\cdot(-s^2+s^3)\cdot T \end{pmatrix} \cdot \begin{pmatrix} u_{HT} \\ u'_{HT} \\ \dot{u}_{HT} \end{pmatrix} \tag{35}$$

$$\dot{u} = \dot{\mathbf{f}}^T\mathbf{u} = \begin{pmatrix} (2-r)r\cdot(6s-6s^2) \\ (-1+r)Hr\cdot(6s-6s^2) \\ (2-r)\cdot r\cdot(-2s+3s)\cdot T \end{pmatrix} \cdot \begin{pmatrix} u_{HT} \\ u'_{HT} \\ \dot{u}_{HT} \end{pmatrix} \tag{36}$$

Top load σ_H and top load vector **b**:

$$\sigma_H = f_H\sigma_{HT} = (3s^2-2s^3)\sigma_{HT} \tag{37}$$

$$\mathbf{b} = \int_0^1 \mathbf{f}_H \sigma_H T \, d s \qquad \mathbf{f}_H = \begin{pmatrix} 3 s^2 - 2 s^3 \\ 0 \\ \left(-s^2 + s^3\right) \cdot T \end{pmatrix} \tag{38}$$

Principle:

$$\begin{aligned} &\delta \mathbf{u}^T E \int_0^T \int_0^H \mathbf{f}' \mathbf{f}'^T d y \, d t \, \mathbf{u} + \delta \mathbf{u}^T \frac{1}{2k} \int_0^T \int_0^H \dot{\mathbf{f}} \mathbf{f}^T d y \, d t \, \mathbf{u} + \ldots \\ &\qquad \ldots + \delta \mathbf{u}^T \frac{1}{2k} \int_0^T \int_0^H \mathbf{f} \dot{\mathbf{f}}^T d y \, d t \, \mathbf{u} - \delta \mathbf{u}^T \int_0^T \mathbf{f}_H \sigma_H d t = 0 \end{aligned} \tag{39}$$

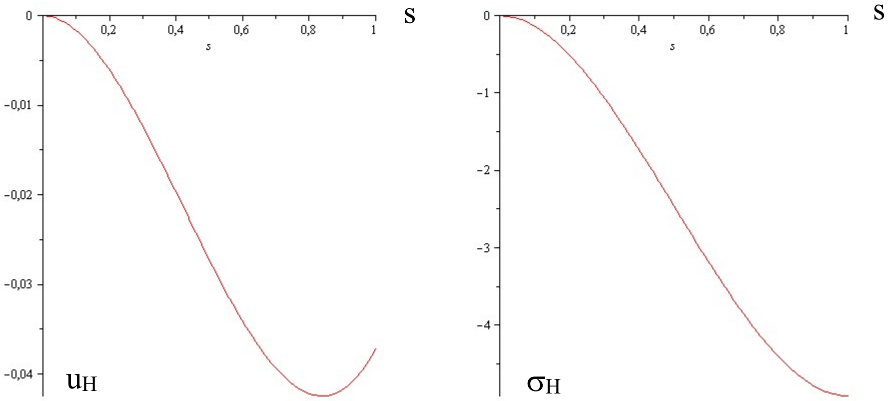


Figure 8: Displacement u_H and stress σ_H at the top.

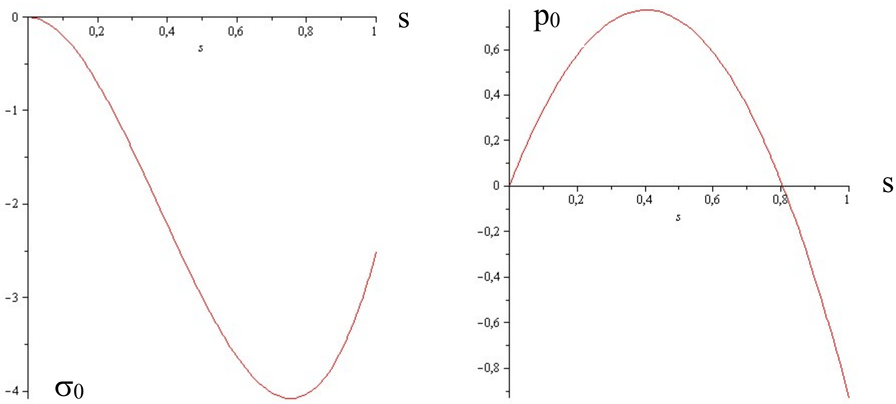


Figure 9: Stress σ_0 and pressure p_0 the bottom.

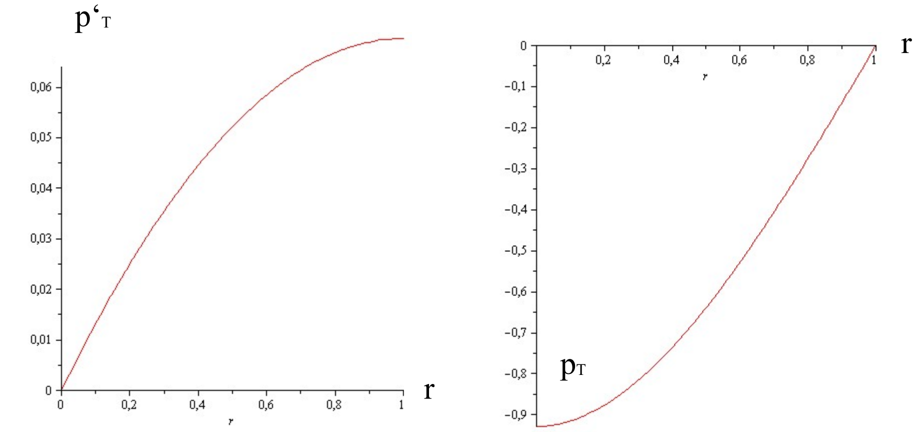


Figure 10: Pressure gradient p_T' and pressure p_T at the end of duration T.

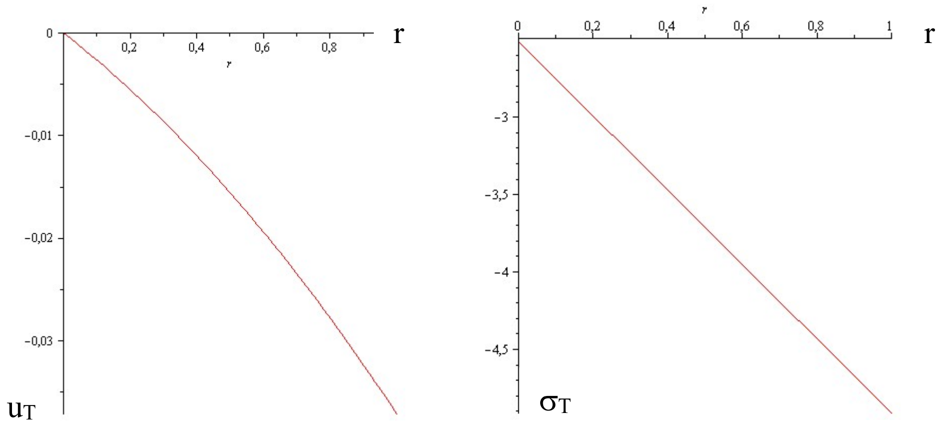


Figure 11: Displacement u_T and stress σ_T at the end of duration T.

The most obvious space time effect is the slightly decreasing displacement u towards the end of duration T. The pressure p turns to tension. This effect vanishes if the time derivation at the end is set to zero or if the duration T is extended towards infinity.

$$u_{HT} = \frac{3\,H\,T\,k\left(2080\,T\,E\,k + 1127\,H^2\right)\sigma_{HT}}{7826\,H^2 T\,E\,k + 315\,H^4 + 3120\,T^2 E^2} \tag{40}$$

$$u_{HT} = -0,03712 \tag{41}$$

$T \rightarrow \infty$

$$u_{HT} = \frac{3 \cdot 2080\,H\,\sigma_{HT}}{2 \cdot 3120\,E} \tag{42}$$

$$u_{H\infty} = -0,05 \tag{43}$$

CONCLUSIONS

- An equal treatment of space and time in connection with low order approximation is an appropriate approach to analyze an elastic stripe subjected to the extrusion of a liquid due to compression.
- The minimization of virtual worktime is a powerful extension of the minimization of virtual work.

REFERENCES

[1] KJ. Bathe, *Finite-Elemente-Methoden,* Springer-Verlag, 2012.
[2] R.M. Bowen: Compressible porous media models by use of the theory of mixtures, International Journal of Engineering Sciences, Elsevier, Amsterdam, 1982
[3] M. Göttlicher, Hybrid Analysis for Continua with Solid and Liquid Properties in Infinite High Tubes. VIII International Conference on Computational Methods for Coupled Problems in Science and Engineering, Barcelona, 2019.
[4] Schwarz, A., Averweg, S., Bluhm, J., Schröder J.: A mixed least-squares finite element formulation within the framework of the theory of porous media, Pamm, Wiley-VCH Verlag, Weinheim, 2019.
[5] O.C. Zienkiewicz and R.C. Taylor, *The finite element method*, 6th Edition, Elsevier, 2005.