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MIXED FINITE ELEMENT FORMULATION FOR LAMINATED COMPOSITE CYLINDRICAL SHELLS BASED ON REFINED ZIGZAG THEORY

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Summary. This paper presents a mixed finite element formulation to examine the linear static behavior of thin and moderately thick laminated composite cylindrical shells within the framework of the Refined Zigzag Theory (RZT). The RZT is very suitable for modeling thick and highly heterogeneous laminated composite structures without the need for the shear correction factor. The system's stationary condition is expressed by using the Hellinger-Reissner principle. Finite element model employs four-noded quadrilateral elements with bilinear shape functions, meeting the C0 continuity requirements. The mixed finite element equations produce direct nodal displacements and stress resultants simultaneously. Comparisons and convergence analyses are performed by considering various lamination configurations and boundary conditions for validation purposes.

1 INTRODUCTION

Laminated composite shell structures plays a central role in real-world engineering, such as aircraft fuselages, car body panels, orthopedic implants, and pressure vessels in the form of beam and plate structures. Laminated composites must be thoroughly examined under various loads during design to identify stress concentrations and ensure structural integrity. Due to the time and cost inefficiencies of experimental investigations, reliable computational modeling methods are certainly necessary for analyzing the mechanical behavior of composite structures. Theoretical approaches are key in assessing structural systems, offering a comprehensive view across various scenarios. Many studies in the literature examine and aim to enhance the accuracy of mechanical analyses for laminated structures. Notable theories in this field include Equivalent Single Layer (ESL) Theories, Layer-wise (LW) Theories, and Zigzag (ZZ) Theories (Figure 1). According to ESL, displacements are calculated on a single equivalent layer, regardless of the number of material layers. This makes ESL extremely efficient in terms of calculation time. On the other hand, LW theories have higher accuracy and longer calculation time. ZZ theories aim to describe in-plane deformations more realistically by defining additional cross-section distortion functions. The number of kinematic variables is independent of the number of material layers. Therefore, ZZ theories provide high accuracy, as in LW, and with high computational efficiency, as in ESL which can be concluded as an optimum theory between LW and ESL. The subcategories of the ESL, some LW, and ZZ can be found in Table 1. Recently, Bab and Kutlu [1] summarized the researchers who worked on static or dynamic analysis of laminated composite shells applying FSDT or HSDT which are the subcategories of ESL Theories.

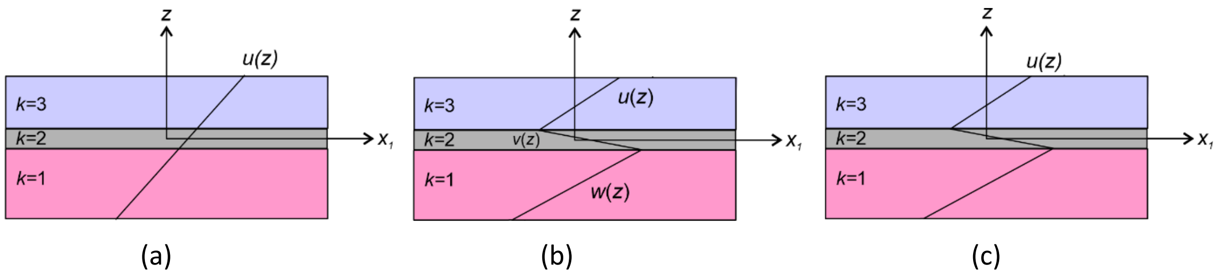

Figure 1: Existing structural models (a) Equivalent Single-Layer (ESL) Theories (b) Layer-wise Theories (LW) (c) Zigzag (ZZ) Theories

Presenting a simplified layerwise higher-order zigzag model for bending laminated composite shells, Xavier et al. [2] ensure a cubic variation of in-plane displacements and transverse shear stresses. This model eliminates the need for shear correction factors, maintaining variable consistency with the first-order shear deformation shell theory, regardless of the layers involved. Dumir et al. [3] proposed an improved efficient zigzag theory (IZIGT) and an improved third-order theory (ITOT) for the static analysis of laminated and sandwich circular shell structures subjected to thermal loads. In ITOT, inplane displacements demonstrate a global cubic variation throughout the thickness, while in IZIGT, they are simulated with an extra layerwise zigzag linear variation. Yasin and Kapuria [4] employed the Zigzag (ZZ) methodology to examine the static and dynamic characteristics of singly- and doubly-curved composite and sandwich shallow shells. Exploring the natural vibration response of laminated and sandwich shells, Kumar et al. [5] employed a 2D finite element model based on a higher order zigzag theory (HOZT) addressing cross curvature effects using Sanders' approximations and ensuring inter-laminar shear stress continuity. Nath and Das [6] introduce a precise zigzag theory for analyzing the static and free vibration behavior of functionally graded (FG) cylindrical shells and rectangular plates. Accurate results are obtained for displacement, stress, and natural frequencies, considering various parameters by representing in-plane displacements through a combination of linear layerwise and cubic global terms. Gupta and Pradyumna [7] extend the third-order shear deformation theory, incorporating zig-zag effects using the Murakami zigzag function, for linear and nonlinear static analyses of spherical, cylindrical, and hyperboloid shells. Carrera [8] mentioned and reviewed 3 independent multilayered Zigzag shell theories which are Lekhnitskii, Ambartsumian and Reissner. Recently, Magisano et al. [9] investigated laminated hemispherical shells according to two different warping profile: the first with independent transverse shear deformations and the second is with a single zigzag shape.

Table 1: Existing Structural Models

Equivalent Single Layer (ESL) Theories	Layer-wise Theories (LW)	Zigzag (ZZ) Theories
Classical Laminate Theory (CLT)	Carrera Unified Formulation (CUF)	Di Sciuva’s Zigzag Theory
First-order Shear Deformation Theory (FSDT)	Generalized Unified Formulation (GUF)	Murakami’s Zigzag Theory
Higher-order Shear Deformation Theory(HSDT)	Reddy’s Generalized LW Theory	Averill’s Zigzag Theory
	Ferreira’s Linear LW Theory	İcardi’s Zigzag Theory
	Non-linear LW Theory	Refined Zigzag Theory

Versino et al. [10] presented an extended version of Tessler's Refined Zigzag Theory (RZT) for beams and plates, applicable to double-curved multilayer shells. The method integrates Naghdi's shell model with RZT kinematics, utilizing Assumed Natural Strain and Enhanced Assumed Strain methodologies. Gherlone et al. [11] proposed C0-continuous flat shell elements based on RZT for static and free vibration analyses of curved composite and sandwich structures. These elements address shear-locking, incorporate essential degrees of freedom, and include zigzag rotations that compute normal distortion. Zhang et al. [12] suggested an RZT-based composite laminated model for free vibration and buckling analysis of cylindrical and spherical shells with various lamination schemes. Utilizing piecewise-linear zigzag functions, the study offers a realistic representation of transverse shear-flexible shells and higher-up computational efficiency compared to high-order models. Gao et al. [13] proposed an advanced bending model for composite laminated shells based on the RZT with in-plane linear zigzag functions, eliminating the need for shear correction coefficients. In their study, displacement and the strain terms are specified according to Reddy’s assumption [14]. Gao et al. [15] also combined Reissner’s Mixed Variational Theorem (RMVT) with RZT to enhance the accuracy and efficiency of the analyses by implementing additional curvature terms which do not exist in Reddy’s assumptions. Layer-wise (LW) theories are divided into three categories: Displacement-based, Mixed, and Non-linear. According to Table 1, CUF [16] and GUF [17] are part of Mixed LW, while Reddy's Generalized LW [14] and Ferreira's Linear LW [18] correspond to Displacement-based LW. The details about LW Theories can be found in Liew et al. [19]. Recently, Kutlu [20] and Kutlu et al. [21] employed the mixed finite element method to perform the RZT-based static analysis of laminated composite beams [22] and plates, respectively. In this paper, the stress behavior of laminated composite shells is investigated for the first time using a combination of the mixed finite element method [23,24] and RZT. The radius terms of the strain components are employed for the first time in the same manner as in the FSDT, since RZT is an extension of FSDT, and the radius terms are mostly related to the first terms, which are the same in both FSDT and RZT. By employing the (RZT), the necessity for the shear correction factor in FSDT is eliminated. Utilizing the Hellinger-Reissner principle, the stationary condition of the functional for the system is represented. Finite element discretization employs four-noded quadrilateral two-dimensional elements and the bilinear shape functions are utilized due to C0 continuity requirements of the formulation. Upon solving the mixed finite element equations, displacements, and stress resultants are directly obtained at the nodes. To affirm the efficacy of the presented solution method, analyses of comparison and convergence are conducted across diverse lamination schemes and under various boundary conditions.

2 KINETIC RELATIONS OF RZT CYLINDRICAL SHELL

According to the RZT, the displacement fields in the k 'th layer of the laminated composite cylindrical shell (Figure 1) are described as follows:

$$u^{(k)}\left(x_1,x_2,z,t\right)=u\left(x_1,x_2,t\right)+z\theta_1\left(x_1,x_2,t\right)+\phi_1^{(k)}(z)\psi_1(x_1,x_2,t) \tag{1}$$
$$v^{(k)}\left(x_1,x_2,z,t\right)=(1+\frac{z}{R_2})v\left(x_1,x_2,t\right)+z\theta_2\left(x_1,x_2,t\right)+\phi_2^{(k)}(z)\psi_2(x_1,x_2,t)$$
$$w^{(k)}\left(x_1,x_2,z,t\right)=w\left(x_1,x_2,t\right)$$

Here, $u^{(k)}\left(x_1,x_2,z,t\right)$ and $v^{(k)}\left(x_1,x_2,z,t\right)$ represent axial displacement field and $w^{(k)}\left(x_1,x_2,z,t\right)$ represent transverse displacement field of the shell. $u\left(x_1,x_2,t\right)$ and $v\left(x_1,x_2,t\right)$ are the axial displacements and $w\left(x_1,x_2,t\right)$ is the deflection of the shell at its mid-surface where x_1 and x_2 are curvilinear orthogonal coordinates defined at the mid-surface of the shell. z is the coordinate in the direction of shell thickness (Figure 2). R_2 is the radius of curvature of the cylindrical shell.

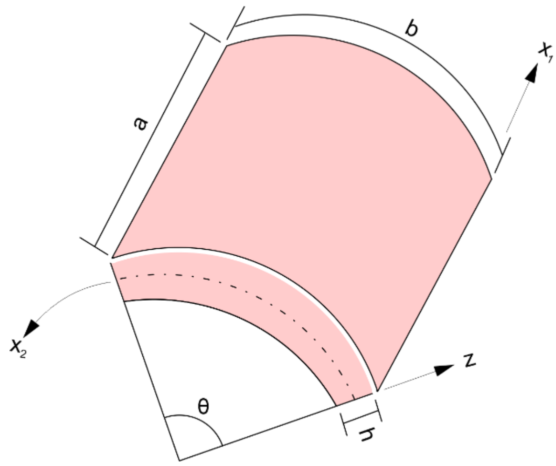

Figure 2: Cylindrical shell coordinate system

Also, $\theta_1(x_1, x_2, t)$ depicts the rotation of the section about the x_2-axis and $\theta_2(x_1, x_2, t)$ depicts the rotation of the section about the x_1-axis. $\psi_1(x_1, x_2, t)$ and $\psi_2(x_1, x_2, t)$ denote zigzag parameters, while $\phi_1^{(k)}(z)$ and $\phi_2^{(k)}(z)$ are denoting linear zigzag functions along x_1 and x_2 axis, respectively and can be expressed as below:

$$\phi_1^{(k)}(\zeta) = \frac{1}{2}(1-\zeta)u_{(k-1)} + \frac{1}{2}(1+\zeta)u_{(k)}$$
$$\phi_2^{(k)}(\zeta) = \frac{1}{2}(1-\zeta)v_{(k-1)} + \frac{1}{2}(1+\zeta)v_{(k)} \tag{2}$$

The local variable $\zeta_{(k)} \in [-1,1]$ can be specified as:

$$\zeta_{(k)} = \frac{2z - z_{(k)} - z_{(k-1)}}{z_{(k)} - z_{(k-1)}} \tag{3}$$

Where $\xi_{k-1} \leqslant \xi \leqslant \xi_k$.

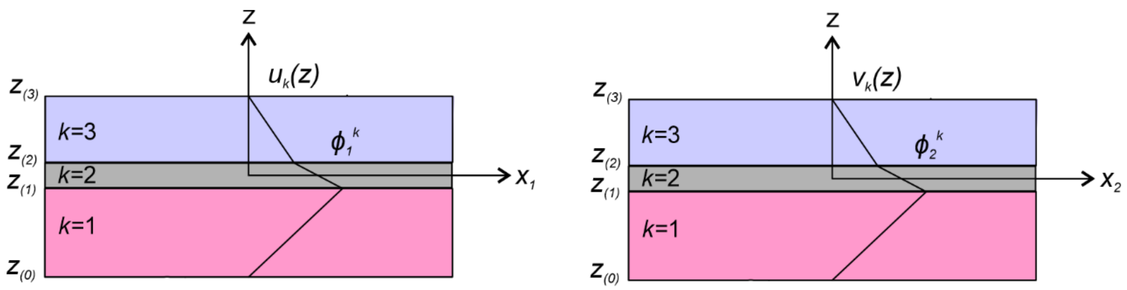

Figure 3: Zigzag functions of the RZT through thickness

The unknown displacements $u_{(k)}$ and $v_{(k)}$ (see Eqn. (2)) at the interfaces of each layer vanish at the top $(u_{(3)} = v_{(3)} = 0)$ and the bottom $(u_{(0)} = v_{(0)} = 0)$ of the cross-section as it can be seen from Figure 3. The interface displacements are recursively acquired by taking the derivative of the zigzag functions $\beta_1^{(k)} \equiv \phi_{1,z}^{(k)}$ and $\beta_2^{(k)} \equiv \phi_{2,z}^{(k)}$ in the form:

$$u_{(k)} = 2h^{(k)}\beta_1^{(k)} + u_{(k-1)}$$
$$v_{(k)} = 2h^{(k)}\beta_2^{(k)} + v_{(k-1)} \tag{4}$$

The derivatives of the zigzag functions exhibit piece-wise constancy and experience discontinuities at the interfaces between layers. These values can vary across layers, reflecting differences in material composition, and can be indicated as:

$$\beta_1^{(k)} \equiv \frac{G_1}{\bar{Q}_{44}^{(k)}} - 1 \quad , \quad \beta_2^{(k)} \equiv \frac{G_2}{\bar{Q}_{55}^{(k)}} - 1 \quad , \quad (k = 1, 2, ..., N-1, N) \tag{5}$$

Where $\bar{Q}_{44}^{(k)}$ and $\bar{Q}_{55}^{(k)}$ correspond the transverse shear moduli of the kth layer while G_1 and G_2 correspond the transverse shear rigidity and can be indicated as:

$$G_1 = \left(\frac{1}{2h} \sum_{k=1}^{N} \frac{2h^{(k)}}{\bar{Q}_{44}^{(k)}} \right)^{-1} \quad , \quad G_2 = \left(\frac{1}{2h} \sum_{k=1}^{N} \frac{2h^{(k)}}{\bar{Q}_{55}^{(k)}} \right)^{-1} \tag{6}$$

2.1 Strain Components of RZT

By the linear relation between displacement and strain, the strain fields in the kth layer using RZT can be represented in relation to the displacement components as follows:

$$\begin{aligned}
\varepsilon_{11} &= u_{,1} + z\theta_{1,1} + \phi_1 \psi_{1,1} + \frac{w}{R_1} \\
\varepsilon_{22} &= v_{,2} + z\theta_{2,2} + \phi_2 \psi_{2,2} + \frac{w}{R_2} \\
\gamma_{1z} &= w_{,1} + \theta_1 + \phi_{1,z} \psi_1 - \frac{u}{R_1} \\
\gamma_{2z} &= w_{,2} + \theta_2 + \phi_{2,z} \psi_2 - \frac{v}{R_2} \\
\gamma_{12} &= u_{,2} + v_{,1} + z\left(\theta_{1,2} + \theta_{2,1}\right) + \phi_1 \psi_{1,2} + \phi_2 \psi_{2,1} + z\,\tfrac{1}{2}\left(\frac{1}{R_2} - \frac{1}{R_1} \right)\left(v_{,1} - u_{,2} \right)
\end{aligned} \tag{7}$$

2.2 Equilibrium Equations

The equilibrium equations of the laminated shell based on the RZT can be formulated as follows:

$$\begin{aligned}
&\delta u: N_{11,1} + N_{12,2} - \tfrac{1}{2}\left(\frac{1}{R_2} - \frac{1}{R_1} \right) M_{12,2} + \frac{Q_1}{R_1} = 0 \\
&\delta v: N_{12,1} + N_{22,2} + \tfrac{1}{2}\left(\frac{1}{R_2} - \frac{1}{R_1} \right) M_{12,1} + \frac{Q_2}{R_2} = 0 \\
&\delta w: Q_{1,1} + Q_{2,2} + p - \left(\frac{N_{11}}{R_1} + \frac{N_{22}}{R_2} \right) = 0 \\
&\delta\theta_1: M_{11,1} + M_{12,2} - Q_1 = 0 \\
&\delta\theta_2: M_{12,1} + M_{22,2} - Q_2 = 0 \\
&\delta\hat{\psi}_1: \hat{M}_{11,1} + \hat{M}_{12,2} - \hat{Q}_1 = 0 \\
&\delta\hat{\psi}_2: \hat{M}_{21,1} + \hat{M}_{22,2} - \hat{Q}_2 = 0
\end{aligned} \tag{8}$$

Here, $N_{\alpha\beta}$ corresponds to in-plane stress resultants, $M_{\alpha\beta}, \hat{M}_{\alpha\beta}$ denotes bending stress resultants, and $Q_\alpha, \hat{Q}_\alpha$ represents transverse shear stress resultants where $\alpha, \beta = 1, 2$, while $p(x_1, x_2)$ signifies the applied transverse distributed load.

These resultants are interconnected with the kinematic variables based on the properties of the ply as:

$$\mathbf{P} = \mathbf{E}\mathbf{e}^{\mathbf{u}} \quad or \quad \begin{Bmatrix} \mathbf{N}_m \\ \mathbf{M}_b \\ \mathbf{Q}_s \end{Bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{0} \\ \mathbf{B}^T & \mathbf{D} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{G} \end{bmatrix} \begin{Bmatrix} \mathbf{e}_m \\ \mathbf{e}_b \\ \mathbf{e}_s \end{Bmatrix} \tag{9}$$

In equation (9), $\mathbf{P}$ represents the stress resultants (integrals of stresses and their moments along the thickness direction), $\mathbf{e}^{\mathbf{u}}$ presents strain measures, and $\mathbf{E}$ corresponds to the sectional stiffness matrix, and its terms are calculated in the same way as for the RZT of laminated composite plates and can be found in Kutlu et al. [21]. The stress $\mathbf{N}_m$, moment $\mathbf{M}_b$, and shear resultants $\mathbf{Q}_b$ and the compliance matrix ($\mathbf{S} = \mathbf{E}^{-1}$) are also described in Kutlu et al. [21].

Furthermore, the kinematic variables can be represented in the following manner:

$$\begin{aligned}
\mathbf{e}_m^T &= \left\{ u_{,1} + \frac{w}{R_1} \quad v_{,2} + \frac{w}{R_2} \quad u_{,2} + v_{,1} \right\} \\
\mathbf{e}_b^T &= \left\{ \theta_{1,1} \quad \psi_{1,1} \quad \theta_{2,2} \quad \psi_{2,2} \quad \theta_{1,2} + \theta_{2,1} + \tfrac{1}{2}\left(\frac{1}{R_2} - \frac{1}{R_1} \right)\left(v_{,1} - u_{,2} \right) \quad \psi_{1,2} \quad \psi_{2,1} \right\} \\
\mathbf{e}_s^T &= \left\{ w_{,2} + \theta_2 - \frac{v}{R_2} \quad \psi_2 \quad w_{,1} + \theta_1 - \frac{u}{R_1} \quad \psi_1 \right\}
\end{aligned} \tag{10}$$

2.4 Hellinger-Reissner Principle and Functional

In equation (11), Hellinger-Reissner functional is integrated over area and applied to shell structures. The stress resultants and strain measures replace the individual stress and strain components as:

$$\delta \Pi_{HR} = \int_A \left(\mathbf{e}^{\mathbf{u}} - \mathbf{e}^{\mathbf{P}} \right)^T \delta \mathbf{P} \, \mathrm{d}A + \int_A \left(\mathbf{P}^T \delta \mathbf{e}^{\mathbf{u}} - \mathbf{p}^T \delta \mathbf{u} \right) \mathrm{d}A - \int_\Gamma \hat{\mathbf{t}}^T \delta \mathbf{u} \, \mathrm{d}\Gamma = 0 \tag{11}$$

In equation (11), the variable A represents the shell's mid-plane, $\mathbf{P}$ encompasses the stress resultants, and $\mathbf{p}$ refers to the transverse load applied to the shell.

Ultimately, by utilizing equations above, the detailed expression for the first variation of the functional derived for a laminated composite shell can be formulated. This equation incorporates both displacement and force-moment field variables. In the expression for the first variation of the functional, there are only first-order derivatives. In this way, the shape functions used in the finite element discretization can have C^0 continuity.

2.4 Finite Element Discretization

As a further step, finite element discretization employs two-dimensional four-noded quadrilateral elements along with bi-linear shape functions. The details about the shape functions, the system matrix, the unknown vector, and the post processing can be found in Kutlu et al. [21]. By solving the system equation, both the displacement-type and stress resultant-type field variables for the laminated composite shell are directly obtained at the nodes. Consequently, strain measures can be computed through matrix operations without needing any numerical differentiation. Subsequently, kinematic and constitutive relations are employed to determine the distribution of in-plane stress components along the cross-section at the nodes.

3 NUMERICAL EXAMPLES

The mixed finite element formulation proposed for laminated composite cylindrical shells is named MRZT-S (Mixed Refined Zigzag Theory for Shells), and a series of numerical examples are discussed to demonstrate its performance and effectiveness in stress resultants and displacement calculations under static loading, and the results are evaluated. The solutions are produced through a Fortran-based program developed by the authors. By making comparisons with Khdeir et al. [25], who analytically solved Higher Order Shear Theory (HSDT), Bab and Kutlu [1], who numerically solved HSDT, Asadi et al. [26], who made FSDT and 3D finite element analysis solutions, analytical Higher-order Zigzag Theory (HOZT) solutions of Kumar et al. [27] in the literature; the accuracy of the formulation and the developed program is tested. Various element meshes as $n_e = 8\times8$, 16×16, and 20×20 were used in the examples. Solutions for different loading and support conditions were presented. Two types of materials (A: ($E_1 = 132379340029, E_2 = E_3 = 10755821377$, $G_{12} = G_{13} = 5653700980.4, G_{23} = 3605958064.3$, $\nu_{12} = \nu_{13} = 0.24$, $\nu_{23} = 0.49$)[GPa] B: $E_1/E_2 = 25, E_1/E_3 = 25, G_{12} = G_{13} = 0.5E_2, G_{23} = 0.2E_2$, $\nu_{12} = \nu_{13} = \nu_{23} = 0.25$, $E_2 = 1$GPa) are used.

3.1 [0/90/0] Cylindrical Shell with Various Boundary conditions under Sinusoidal Load

A symmetrically laminated cylindrical shell is analyzed using the proposed mixed finite element formulation and its results are compared with the literature. The shell is under the influence of transverse load of function $p(x_1, x_2) = p_0 \sin(\pi x_1/a)\sin(\pi x_2/b)$ distributed over its outer surface. Here a and b are the arc-lengths of the shell in the x_1 and x_2 axes directions, respectively. The curvature in the cylindrical shell is on the x_2 axis, so $R_{x_1} = \infty$. The cross section is defined with the [0/90/0] material orientation, and each layer of equal thickness is considered to be formed with material A. The $\rho = a/h = 10$ value is considered as the shell length-thickness ratio. In Table 2, the SSSS boundary condition (BC) indicates that all four edges of the shell are simply supported. According to SCSC boundary condition, the two edges of the shell in the x_1-axis direction are simply supported, while the other two edges in the x_2-axis direction are clamped. To evaluate the results presented in Table 2 in a general structure, the non-dimensionalization process is carried out according to Equation (12):

Table 2: Convergence of laminated symmetrical cylindrical shell deflection under sinusoidal loading

BC'S	R/a	MRZT-S	MHST	Khdeir et al. (Analytical HSDT)	Difference (%)	
		8×8	8×8		MRZT-S	MHST
SSSS	5	0.9392	0.9354	0.9524	1.38	1.78
	10	0.9503	0.9473	0.9644	1.47	1.77
	50	0.9538	0.9512	0.9683	1.49	1.77
SCSC	5	0.4410	0.4178	0.4495	1.90	7.05
	10	0.4435	0.4203	0.4523	1.95	7.08
	50	0.443	0.4211	0.4532	1.96	7.09

According to Table 2, the MRZT-S results are in good agreement with the analytical results and provide more converged values than the MHST. The percentage difference is smaller when the boundary condition is fully simply supported.

$$\bar{w} = \frac{10^2 E_2 h^3 w(a/2, b/2)}{p_0 a^4} \tag{12}$$

3.2 [0/90/0] Simply Supported Cylindrical Shell under Uniform Load

Uniformly $p(x) = p_0$ loaded simply supported cylindrical shells with material B [0/90/0] layouts and are discussed in this section. The results obtained with the element meshes of 8×8, 16×16 and 20×20 for various thickness ratios ($\rho = a/h = 10, 20$) are compared with the FSDT and ANSYS 3D finite element solutions of Asadi et al. [26]. The non-dimensionalization of the values calculated as follows:

$$\bar{w} = \frac{10^3 E_2 h^3 w(a/2, b/2)}{p_0 a^4}, \quad \overline{M} = \frac{10^3 M(a/2, b/2)}{p_0 a^2}, \quad \overline{N} = \frac{10^3 N(a/2, b/2)}{p_0 a^3} \tag{13}$$

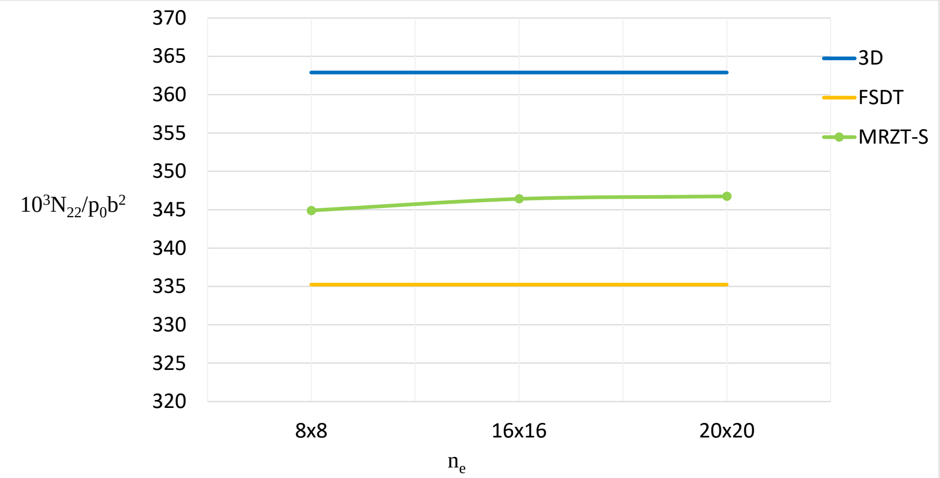

Figure 4: Dimensionless normal force values in a simply supported laminated composite shell under uniform load $(R/a = 1, \rho = 10)$

Figure 4 shows that the results obtained from the current formulation (MRZT-S) are highly compatible with the 3D solution. MRZT-S results also exhibit a consistent convergence behavior to the 3D values and divergent behavior to the FSDT depending on the increasing number of elements.

5 CONCLUSION

This study investigates the displacement and stress distributions in laminated composite cylindrical shells under various boundary and loading conditions using a mixed finite element method based on the Refined Zigzag Theory. The method employs two-dimensional four-noded elements, allowing direct calculation of force, moment, and displacement variables at the nodes. Convergence and comparison analyses show strong agreement with reference analytical, numerical, and 3D solutions. The proposed formulation has potential for future development and application to various problems.

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