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APPROXIMATE CALCULATION OF CHANGES IN LOCAL AND GLOBAL QUANTITIES DUE TO DESIGN MODIFICATIONS

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Summary. This contribution introduces a method for goal-oriented and global reanalysis. It allows the prediction of changes in selected quantities of interest by using the changes in the primal and dual solutions resulting from structural modifications (e.g., changes in shape, topology, or material properties). The approach uses a goal-oriented method that includes both primal and dual problems. In particular, this method is easy to integrate into existing finite element programs because it does not require derivatives with respect to design variables.

1 INTRODUCTION

Design modifications are frequently explored in numerous applications. These design variables may include cross-sectional dimensions, geometric features, topology, or material properties. Typical applications that involve iterative design processes include structural optimization, reliability analysis, and structural damage assessment. In many instances, hundreds or even thousands of different design configurations are examined, requiring the solution of the governing state equation at each design iteration. This repeated structural analysis demands substantial computational resources, particularly for large-scale problems.

An efficient reanalysis method can significantly reduce the overall computational cost. Classical global reanalysis techniques aim to accurately and efficiently estimate changes in state variables due to design modifications without directly solving the modified equation set for the modified problem. The literature has extensively studied reanalysis techniques for calculating changes in state variables due to design modifications, e.g., [2, 6, 9, 15, 5, 12], among many others. It has been formulated for linear statics and dynamic problems. Furthermore, reanalysis methods have been applied to sensitivity analysis and optimization problems [7, 14, 1].

Very often, we are interested only in specific *quantities of interest* J, and so-called *goal-oriented* or *duality techniques* or *adjoint state methods* can be used to compute J. This is known as the concept of influence functions in structural mechanics, see, e.g., [4]. The quantities of interest could be point values, such as a displacement component, a stress component at a point, or an integral value.

This paper presents a method for predicting the change in a quantity of interest ΔJ resulting from structural modifications. The technique calculates changes in the primal and dual solutions using the *residual increment approximation* (RIA) method described in [12]. This reanalysis procedure is based on a formulation using residual increments. Unlike other existing reanalysis methods, such as the CA method (e.g., the CA method [8]), which rely on evaluating changed stiffness matrices, this method only

requires the computation and storage of residual vectors. Therefore, it provides an efficient *goal-oriented reanalysis method* to compute changes in quantities of interest due to given design modifications with sufficient accuracy without directly solving the changed problem's modified equations.

2 FORMULATION OF GOAL-ORIENTED ANALYSIS

2.1 The quantities of interest

In goal-oriented or duality techniques, we are interested in a quantity of interest J. This quantity depends on the state variables $\mathbf{u}$ and certain design variables $\mathbf{s}$, such as parameters describing cross-sections, geometry, topology, or material properties. In this paper, $J(\mathbf{s})(\mathbf{u})$ is linear in $\mathbf{u}$ but possibly nonlinear in $\mathbf{s}$. Moreover, we assume that J is differentiable with respect to $\mathbf{u}$.

The quantities of interest may include point values, such as a displacement component $u_i(\boldsymbol{X})$ or a stress component $\sigma_{ij}(\boldsymbol{X})$ at a point $\boldsymbol{X}$. Additionally, they can be integral values, for example,

$$J(\mathbf{s})(\mathbf{u}) = u_i(\boldsymbol{X}) \quad \text{or} \quad J(\mathbf{s})(\mathbf{u}) = \sigma_{ij}(\boldsymbol{X}) \quad \text{or} \quad J(\mathbf{s})(\mathbf{u}) = \int_A \sigma_{ij} \, \mathrm{d}A. \tag{1}$$

2.2 The primal and dual (adjoint) problems

The state $\mathbf{u}$ is determined by the *primal problem* in terms of a residual $\mathbf{R}(\mathbf{s})(\mathbf{u})$. This paper considers linear problems, i.e., $\mathbf{R}$ is linear in $\mathbf{u}$. For a given fixed design $\mathbf{s}$, the *primal solution* $\mathbf{u}$ is obtained from

$$\mathbf{R}(\mathbf{s})(\mathbf{u}) = \mathbf{K}(\mathbf{s})\,\mathbf{u} - \mathbf{f}(\mathbf{s}) = \mathbf{0}, \tag{2}$$

where $\mathbf{K}$ is the stiffness matrix and $\mathbf{f}$ is the primal load vector. Both, $\mathbf{K}$ and $\mathbf{f}$ depend only on $\mathbf{s}$.

For a chosen $J(\mathbf{s})(\mathbf{u})$, the corresponding *dual or adjoint solution*, *influence function* or *generalized Green's function* $\mathbf{z}$ is determined by the so-called *dual problem* written in terms of the dual residual vector $\mathbf{R}^*(\mathbf{s})(\mathbf{z})$. For a given fixed design $\mathbf{s}$, the dual solution $\mathbf{z}$ is given from

$$\mathbf{R}^*(\mathbf{s})(\mathbf{z}) = \mathbf{K}^T(\mathbf{s})\,\mathbf{z} - \mathbf{j}(\mathbf{s}) = \mathbf{0} \qquad \text{with} \qquad \mathbf{j} := \left(\frac{\partial J}{\partial \mathbf{u}}\right)^T. \tag{3}$$

Here, $\mathbf{j}$ is the so-called dual load vector. For self-adjoint problems, we have $\mathbf{K} = \mathbf{K}^T$.

It is important to note that the residual vectors $\mathbf{R}$ and $\mathbf{R}^*$ and the quantities of interest J are linear in $\mathbf{u}$ and $\mathbf{z}$, respectively, but possibly nonlinear in $\mathbf{s}$.

2.3 Computing the quantity of interest

The quantity of interest J can be computed in two different ways. In the classical approach, the primal problem $\mathbf{R}(\mathbf{s})(\mathbf{u}) = \mathbf{0}$ is solved, and $J(\boldsymbol{s})(\boldsymbol{u})$ can be computed in a post-processing step.

Alternatively, the dual solution $\mathbf{z}$ can also be utilized, which may offer significant advantages.

With the primal problem (2) and the dual problem (3), we have $\mathbf{f} = \mathbf{K}\mathbf{u}$ and $\mathbf{j} = \mathbf{K}^T\mathbf{z}$ respectively, and hence

$$J(\mathbf{s})(\mathbf{u}) = \mathbf{j}^T\,\mathbf{u} = \mathbf{u}^T\,\mathbf{j} = \mathbf{u}^T\,\mathbf{K}^T\,\mathbf{z} = \mathbf{z}^T\,\mathbf{K}\,\mathbf{u} = \mathbf{z}^T\,\mathbf{f} = \mathbf{f}^T\,\mathbf{z}. \tag{4}$$

Finally, for a given fixed design $\mathbf{s}$, the quantity J is given by evaluating a scalar product, i.e.

$$J(\mathbf{s})(\mathbf{u}) = \mathbf{j}^T\,\mathbf{u} \tag{5}$$

or alternatively from

$$J(\mathbf{s})(\mathbf{u}) = \mathbf{f}^T \mathbf{z}. \tag{6}$$

The significant advantage of the formulation (6) is as follows: If we know the dual solution $\mathbf{z}$, we can compute the quantity of interest $J(\mathbf{s})(\mathbf{u})$ for any given primal load vectors $\mathbf{f}$ simply by taking the scalar product of $\mathbf{z}$ and $\mathbf{f}$. Structural mechanics commonly use this method to calculate J for various load cases using influence functions.

3 GOAL-ORIENTED REANALYSIS

The reanalysis process is utilized in various fields that involve making design modifications. Let $\mathbf{s}_0$ be a given initial design and let $\mathbf{u}_0$ be the corresponding solution of (2). Assume a changed design $\mathbf{s}_c = \mathbf{s}_0 + \Delta\mathbf{s}$ and let $\mathbf{u}_c$ be the corresponding solution. Furthermore, let $J(\mathbf{s}_0)(\mathbf{u}_0)$ be the value of J for the initial design $\mathbf{s}_0$ and let $J(\mathbf{s}_c)(\mathbf{u}_c)$ be the value of J for a given changed design $\mathbf{s}_c$.

The general goal-oriented reanalysis problem can be stated as follows: Find the change

$$\Delta J = J(\mathbf{s}_c)(\mathbf{u}_c) - J(\mathbf{s}_0)(\mathbf{u}_0) \tag{7}$$

of J due to given design changes $\Delta \boldsymbol{s}$ with sufficient accuracy without solving the complete modified equations.

3.1 The initial and changed primal and dual problems

The primal and dual problems for the initial and changed designs according to (2) and (3) are given as

$$\mathbf{R}(\mathbf{s}_0)(\mathbf{u}_0) = \mathbf{K}_0\,\mathbf{u}_0 - \mathbf{f}_0 = \mathbf{0} \tag{8}$$

$$\mathbf{R}^*(\mathbf{s}_0)(\mathbf{z}_0) = \mathbf{K}_0^T\,\mathbf{z}_0 - \mathbf{j}_0 = \mathbf{0}. \tag{9}$$

and

$$\mathbf{R}(\mathbf{s}_c)(\mathbf{u}_c) = \mathbf{K}_c\,\mathbf{u}_c - \mathbf{f}_c = \mathbf{0} \tag{10}$$

$$\mathbf{R}^*(\mathbf{s}_c)(\mathbf{z}_c) = \mathbf{K}_c^T\,\mathbf{z}_c - \mathbf{j}_c = \mathbf{0}, \tag{11}$$

where $\mathbf{K}_0 = \mathbf{K}(\mathbf{s}_0)$, $\mathbf{K}_c = \mathbf{K}(\mathbf{s}_c)$, $\mathbf{f}_0 = \mathbf{f}(\mathbf{s}_0)$, $\mathbf{f}_c = \mathbf{f}(\mathbf{s}_c)$, $\mathbf{j}_0 = \mathbf{j}(\mathbf{s}_0)$ and $\mathbf{j}_c = \mathbf{j}(\mathbf{s}_c)$.

3.2 The change in the quantity of interest

The change in the quantity of interest ΔJ can be computed in two different ways. Firstly, we use a formulation with $\mathbf{u}_0$ and the change $\Delta\mathbf{u} = \mathbf{u}_c - \mathbf{u}_0$. With (5) we obtain after some manipulations

$$\begin{aligned}\Delta J &= J(\mathbf{s}_c)(\mathbf{u}_c) - J(\mathbf{s}_0)(\mathbf{u}_0) = \mathbf{j}_c^T\,\mathbf{u}_c - \mathbf{j}_0^T\,\mathbf{u}_0 = \Delta\mathbf{j}^T\,\mathbf{u}_c + \mathbf{j}_0^T\,\Delta\mathbf{u} \\ &= \Delta\mathbf{j}^T\,\mathbf{u}_0 + \mathbf{j}_c^T\,\Delta\mathbf{u}.\end{aligned} \tag{12}$$

The change ΔJ depends on $\Delta\mathbf{j} = \mathbf{j}_c - \mathbf{j}_0$ and the change $\Delta\mathbf{u}$.

Alternatively, the change in J can be expressed in terms of $\mathbf{z}_0$ and the change $\Delta\mathbf{z} = \mathbf{z}_c - \mathbf{z}_0$. By using (6) we obtain

$$\begin{aligned}\Delta J &= J(\mathbf{s}_c)(\mathbf{u}_c) - J(\mathbf{s}_0)(\mathbf{u}_0) = \mathbf{f}_c^T\,\mathbf{z}_c - \mathbf{f}_0^T\,\mathbf{z}_0 = \Delta\mathbf{f}^T\,\mathbf{z}_c + \mathbf{f}_0^T\,\Delta\mathbf{z} \\ &= \Delta\mathbf{f}^T\,\mathbf{z}_0 + \mathbf{f}_c^T\,\Delta\mathbf{z}.\end{aligned} \tag{13}$$

In this formulation, the change ΔJ depends on $\Delta\mathbf{f} = \mathbf{f}_c - \mathbf{f}_0$ and the change $\Delta\mathbf{z}$.

Finally, according to (5) and (6), we obtain the two equivalent relations (12) and (13) to express the change ΔJ.

The significant advantage of the second expression (13) can be summarized as follows: Once the dual solution $\mathbf{z}_0$ and the change $\Delta\mathbf{z}$ for a fixed $\Delta\mathbf{s}$ are determined, the change in the quantity of interest ΔJ can be calculated for any primal load vectors $\mathbf{f}_0$ and corresponding changes $\Delta\mathbf{f}$. In other words, ΔJ can be computed for numerous primal load scenarios simply by evaluating basic scalar products.

4 COMPUTING THE CHANGE IN THE QUANTITY OF INTEREST

To compute ΔJ, we have to evaluate (12) or (13). The changes in the primal and dual loads $\Delta\mathbf{f}$ and $\Delta\mathbf{j}$ can easily be computed with less computational effort. The only challenge is the computation of approximations of the increments $\Delta\mathbf{u}$ or $\Delta\mathbf{z}$ with sufficient accuracy. This can be done by using reanalysis methods. A general efficient reanalysis method for the computation of an approximation of $\Delta\mathbf{u}$ based on residual increment approximations (RIA method) has been presented in [12]. The paper uses this method to compute the changes $\Delta\mathbf{u}$ and $\Delta\mathbf{z}$ with sufficient accuracy without solving the complete modified equations.

4.1 Reanalysis for the primal problem

The general reanalysis problem for the primal solution can be described as follows: Find the change

$$\Delta\mathbf{u} = \mathbf{u}_c - \mathbf{u}_0 \tag{14}$$

resulting from specified design changes $\Delta\boldsymbol{s}$ with adequate accuracy without having to solve the entire modified equations of the primal problem.

The starting point for the reanalysis method is the residual of the changed problem $\mathbf{R}(\mathbf{s}_c)(\mathbf{u}_c)$ defined in (10). Although, the problem is linear in $\mathbf{u}$, the residual is in the general case nonlinear in $\mathbf{s}$, i.e.

$$\mathbf{R}(\mathbf{s}_c)(\mathbf{u}_c) \neq \mathbf{R}(\mathbf{s}_0)(\mathbf{u}_c) + \mathbf{R}(\Delta\mathbf{s})(\mathbf{u}_c). \tag{15}$$

The changed problem (10) can be expressed in terms of the initial design $\mathbf{s}_0$ as

$$\mathbf{R}(\mathbf{s}_c)(\mathbf{u}_c) = \mathbf{R}(\mathbf{s}_0 + \Delta\mathbf{s})(\mathbf{u}_c) = \mathbf{R}(\mathbf{s}_0)(\mathbf{u}_c) + \Delta_s\mathbf{R}(\mathbf{s}_0, \Delta\mathbf{s})(\mathbf{u}_c) = \mathbf{0}. \tag{16}$$

Hence, the residual increment with respect to $\mathbf{s}$ is given as $\Delta_s\mathbf{R}(\mathbf{s}_0, \Delta\mathbf{s})(\mathbf{u}_c) = \mathbf{R}(\mathbf{s}_0 + \Delta\mathbf{s})(\mathbf{u}_c) - \mathbf{R}(\mathbf{s}_0)(\mathbf{u}_c)$. Furthermore, we obtain with $\mathbf{R}(\mathbf{s}_0)(\mathbf{u}_0) = \mathbf{K}_0\mathbf{u}_0 - \mathbf{f}_0 = \mathbf{0}$ for the first term on the right side in (16) the relation

$$\mathbf{R}(\mathbf{s}_0)(\mathbf{u}_c) = \mathbf{R}(\mathbf{s}_0)(\mathbf{u}_0 + \Delta\mathbf{u}) = \mathbf{K}_0\mathbf{u}_0 + \mathbf{K}_0\Delta\mathbf{u} - \mathbf{f}_0 = \mathbf{K}_0\Delta\mathbf{u}. \tag{17}$$

Finally, Eq. (16) leads to

$$\mathbf{K}_0\Delta\mathbf{u} = -\Delta_s\mathbf{R}(\mathbf{s}_0, \Delta\mathbf{s})(\mathbf{u}_c) = -[\mathbf{R}(\mathbf{s}_c)(\mathbf{u}_0 + \Delta\mathbf{u}) - \mathbf{K}_0\Delta\mathbf{u}] =: -\mathbf{Q}(\Delta\mathbf{u}). \tag{18}$$

Note that the residuum $\mathbf{R}(\mathbf{s}_c)(\mathbf{u}_c) = \mathbf{R}(\mathbf{s}_c)(\mathbf{u}_0 + \Delta\mathbf{u})$ is not zero for some approximation of $\Delta\mathbf{u}$. The above equation can be expressed as the recurrence relation

$$\mathbf{K}_0\Delta\mathbf{u}_i = -\mathbf{Q}(\Delta\mathbf{u}_{i-1}), \tag{19}$$

with

$$\begin{aligned}\mathbf{Q}(\Delta\mathbf{u}_{i-1}) &= \mathbf{R}(\mathbf{s}_c)(\mathbf{u}_0 + \Delta\mathbf{u}_{i-1}) - \mathbf{K}_0\Delta\mathbf{u}_{i-1} \\ &= \mathbf{R}(\mathbf{s}_c)(\mathbf{u}_0 + \Delta\mathbf{u}_{i-1}) + \mathbf{Q}(\Delta\mathbf{u}_{i-2}).\end{aligned} \tag{20}$$

Explicitly, we set the initial value $\Delta\mathbf{u}_0 = \mathbf{0}$. Then, recurrence yields the first two values from

$$\Delta\mathbf{u}_1 = -\mathbf{K}_0^{-1}\mathbf{Q}(\Delta\mathbf{u}_0), \qquad \Delta\mathbf{u}_2 = -\mathbf{K}_0^{-1}\mathbf{Q}(\Delta\mathbf{u}_1)$$

with

$$\mathbf{Q}(\Delta\mathbf{u}_0) = \mathbf{R}(\mathbf{s}_c)(\mathbf{u}_0), \qquad \mathbf{Q}(\Delta\mathbf{u}_1) = \mathbf{R}(\mathbf{s}_c)(\mathbf{u}_0 + \Delta\mathbf{u}_1) + \mathbf{Q}(\Delta\mathbf{u}_0).$$

For all other values we have

$$\Delta\mathbf{u}_i = -\mathbf{K}_0^{-1}\mathbf{Q}(\Delta\mathbf{u}_{i-1}) \qquad i = 3, 4, \ldots n. \tag{21}$$

After n iterations, the state $\tilde{\mathbf{u}}_c$ approximation for the changed design $\mathbf{s}_c$ is obtained as $\tilde{\mathbf{u}}_c = \mathbf{u}_0 + \Delta\mathbf{u}_n$.

The stiffness matrix $\mathbf{K}_0$ is the same as used for the solution of the initial design and, therefore, is usually already given in the decomposed form. Therefore, the computation of $\Delta\mathbf{u}_i$ in (19) requires just forward and backward substitution. Only residual vectors must be computed and stored using the reanalysis method.

The reanalysis procedure adapted from Eq. 19 is a local approximation based on information calculated at a single point $(\mathbf{s}_0, \mathbf{u}_0)$. The results can be improved using a vector-valued rational approximation method introduced in [13] and applied to linear reanalysis problems in [15]. This method is used within the numerical examples in the present paper. Details about this method and the overall algorithm of the reanalysis method are given in [12].

4.2 Reanalysis for the dual problem

In the same way, as for the primal problem, we can formulate a reanalysis method for the dual problem, i.e., we want to compute an approximation of $\Delta\mathbf{z}$ to evaluate the relation (13).

The general reanalysis problem for the dual solution is given as follows: Find the change

$$\Delta\mathbf{z} = \mathbf{z}_c - \mathbf{z}_0 \tag{22}$$

as a result of given design changes $\Delta\boldsymbol{s}$ accurately, without having to solve the complete modified equations of the dual problem.

The starting point for the reanalysis method is the residual of the changed problem (11). It can be expressed in terms of the initial design $\mathbf{s}_0$ as

$$\mathbf{R}^*(\mathbf{s}_c)(\mathbf{z}_c) = \mathbf{R}^*(\mathbf{s}_0 + \Delta\mathbf{s})(\mathbf{z}_c) = \mathbf{R}^*(\mathbf{s}_0)(\mathbf{z}_c) + \Delta_s\mathbf{R}^*(\mathbf{s}_0, \Delta\mathbf{s})(\mathbf{z}_c) = \mathbf{0}. \tag{23}$$

Hence, the residual increment with respect to $\mathbf{s}$ is given as $\Delta_s\mathbf{R}^*(\mathbf{s}_0, \Delta\mathbf{s})(\mathbf{z}_c) = \mathbf{R}^*(\mathbf{s}_0 + \Delta\mathbf{s})(\mathbf{z}_c) - \mathbf{R}^*(\mathbf{s}_0)(\mathbf{z}_c)$. Furthermore, we obtain with $\mathbf{R}^*(\mathbf{s}_0)(\mathbf{z}_0) = \mathbf{K}_0^T\mathbf{z}_0 - \mathbf{j}_0 = \mathbf{0}$ for the first term on the right side in (23) the relation

$$\mathbf{R}^*(\mathbf{s}_0)(\mathbf{z}_c) = \mathbf{R}^*(\mathbf{s}_0)(\mathbf{z}_0 + \Delta\mathbf{z}) = \mathbf{K}_0^T\mathbf{z}_0 + \mathbf{K}_0^T\Delta\mathbf{z} - \mathbf{j}_0 = \mathbf{K}_0^T\Delta\mathbf{z}. \tag{24}$$

Finally, Eq. (23) leads to

$$\mathbf{K}_0^T \Delta\mathbf{z} = -\Delta_s \mathbf{R}^*(\mathbf{s}_0, \Delta\mathbf{s})(\mathbf{z}_c) = -[\mathbf{R}^*(\mathbf{s}_c)(\mathbf{z}_0 + \Delta\mathbf{z}) - \mathbf{K}_0^T \Delta\mathbf{z}] =: -\mathbf{Q}^*(\Delta\mathbf{z}). \tag{25}$$

Note that the residuum $\mathbf{R}^*(\mathbf{s}_c)(\mathbf{z}_c) = \mathbf{R}^*(\mathbf{s}_c)(\mathbf{z}_0 + \Delta\mathbf{z})$ is not zero for some approximation of $\Delta\mathbf{z}$. Finally, this can be expressed as the recurrence relation in the form of

$$\mathbf{K}_0^T \Delta\mathbf{z}_i = -\mathbf{Q}^*(\Delta\mathbf{z}_{i-1}) \tag{26}$$

with

$$\begin{aligned} \mathbf{Q}^*(\Delta\mathbf{z}_{i-1}) &= \mathbf{R}^*(\mathbf{s}_c)(\mathbf{z}_0 + \Delta\mathbf{z}_{i-1}) - \mathbf{K}_0^T \Delta\mathbf{z}_{i-1} \\ &= \mathbf{R}^*(\mathbf{s}_c)(\mathbf{z}_0 + \Delta\mathbf{z}_{i-1}) + \mathbf{Q}^*(\Delta\mathbf{z}_{i-2}). \end{aligned} \tag{27}$$

Explicitly, we set the initial value $\Delta\mathbf{z}_0 = \mathbf{0}$. Then, recurrence yields the first two values from

$$\Delta\mathbf{z}_1 = -\mathbf{K}_0^{-T}\mathbf{Q}^*(\Delta\mathbf{z}_0), \qquad \Delta\mathbf{z}_2 = -\mathbf{K}_0^{-T}\mathbf{Q}^*(\Delta\mathbf{z}_1)$$

with

$$\mathbf{Q}^*(\Delta\mathbf{z}_0) = \mathbf{R}^*(\mathbf{s}_c)(\mathbf{z}_0), \qquad \mathbf{Q}^*(\Delta\mathbf{z}_1) = \mathbf{R}^*(\mathbf{s}_c)(\mathbf{z}_0 + \Delta\mathbf{z}_1) + \mathbf{Q}^*(\Delta\mathbf{z}_0).$$

For all other values we have

$$\Delta\mathbf{z}_i = -\mathbf{K}_0^{-T}\mathbf{Q}^*(\Delta\mathbf{z}_{i-1}) \qquad i = 3, 4, \ldots n. \tag{28}$$

After n iterations, the approximation of the dual solution $\tilde{\mathbf{z}}_c$ for the changed design $\mathbf{s}_c$ is obtained as $\tilde{\mathbf{z}}_c = \mathbf{z}_0 + \Delta\mathbf{z}_n$.

In the same way, as for the primal problem, the results are improved by using a vector-valued rational approximation method. This is used within the numerical examples. Details and the overall algorithm are given in [12].

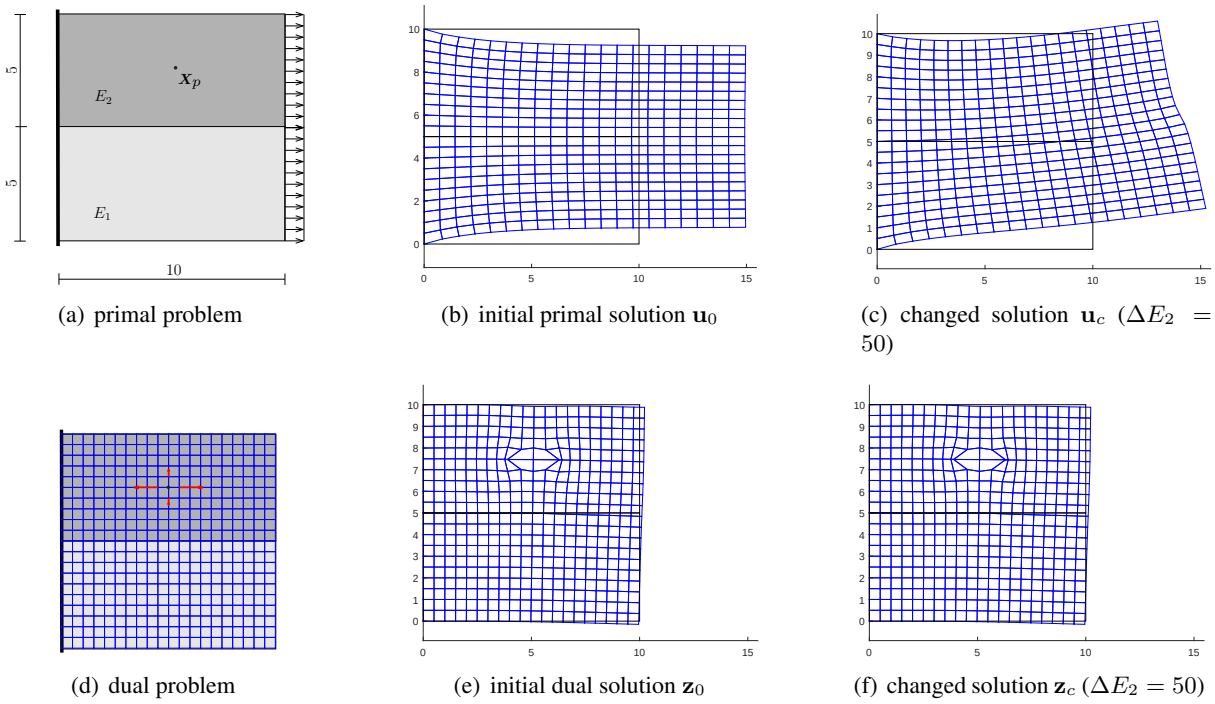
4.3 First-order adjoint sensitivity relation

In numerous applications, the classical *first-order approximation* (FOA) is used to predict the changes in the state variables or quantities of interest due to design modifications, see e.g., [11, 10]. The results are valid only for minimal design changes. This study compares the proposed reanalysis method with the classical FOA for completeness.

To compute the changes in a quantity of interest, so-called adjoint sensitivity analysis can be used; see, e.g., [3]. For a given fixed design change $\Delta\mathbf{s}$ we obtain the first-order approximation $\Delta\tilde{J}$ for the change in the quantity of interest in the form of

$$\Delta\tilde{J} = \left[\frac{\partial J(\mathbf{s}_0)(\mathbf{u}_0)}{\partial\mathbf{s}} - \mathbf{z}_0^T\mathbf{P}_0\right]\Delta\mathbf{s}, \qquad \text{where} \qquad \mathbf{P}_0 = \mathbf{P}(\mathbf{s}_0)(\mathbf{u}_0) = \frac{\partial\mathbf{R}}{\partial\mathbf{s}}(\mathbf{s}_0)(\mathbf{u}_0). \tag{29}$$

The matrix $\mathbf{P}_0$ is the co-called pseudo load matrix, see e.g. [11, 10] for details and explicit formulations. The advantage of the first-order approximation (29) is that the relation depends only on the known initial primal and dual solutions $\mathbf{u}_0$ and $\mathbf{z}_0$, i.e., the changes in the primal and dual solutions are not required. The disadvantage of this method is that it requires the derivatives concerning the design variables, i.e., we have to compute $\frac{\partial J}{\partial\mathbf{s}}$ and $\mathbf{P} = \frac{\partial\mathbf{R}}{\partial\mathbf{s}}$. This can be very difficult and expensive in many situations and model problems. However, it is a robust method and is used in many applications. Therefore, we compare the FOA with the other proposed methods within the numerical examples.



(a) primal problem (b) initial primal solution $\mathbf{u}_0$ (c) changed solution $\mathbf{u}_c$ ($\Delta E_2 = 50$)

(d) dual problem (e) initial dual solution $\mathbf{z}_0$ (f) changed solution $\mathbf{z}_c$ ($\Delta E_2 = 50$)

Figure 1: Bi-material solid under tension: The design variables are Young's moduli E_1 and E_2 in the two domains, i.e., $\mathbf{s} = [E_1 \;\; E_2]^T$ with the initial design $\mathbf{s}_0 = [100 \;\; 100]^T$. A large design change $\Delta\mathbf{s} = [0 \;\; 50]^T$ is investigated. The quantity of interest is the stress component σ_{xx} at point $\boldsymbol{X}_p$, i.e. $J(\boldsymbol{s})(\boldsymbol{u}) = \sigma_{xx}(\boldsymbol{X}_p)$. The dual load case (red arrows in (d)) causes an approximation of a unit dislocation at point $\boldsymbol{X}_p$.

5 NUMERICAL EXAMPLE

In the present paper, we have discussed three methods for approximating the change $\Delta J = J(\mathbf{s}_c)(\mathbf{u}_c) - J(\mathbf{s}_0)(\mathbf{u}_0)$ given a fixed design change $\Delta\mathbf{s}$. These methods are summarized in Table 1 and will be investigated in the following numerical example, which focuses on the model problem of linear elasticity.

Table 1: Summary of different methods to compute approximations of the change ΔJ

Method	Discrete formulation	
1	$\Delta J = \Delta\mathbf{j}^T\,\mathbf{u}_0 + \mathbf{j}_c^T\,\Delta\mathbf{u}$	(see Eq. 12)
2	$\Delta J = \Delta\mathbf{f}^T\,\mathbf{z}_0 + \mathbf{f}_c^T\,\Delta\mathbf{z}$	(see Eq. 13)
3	$\Delta\tilde{J} = \left[\frac{\partial J}{\partial\mathbf{s}} - \mathbf{z}_0^T\mathbf{P}_0\right]\Delta\mathbf{s}$	(see Eq. 29)

Approximations of the changes in the primal and dual solutions $\Delta\mathbf{u}$ and $\Delta\mathbf{z}$ required in methods 1 and 2 are computed by using the reanalysis method from (19) and (26), respectively. Furthermore, the accuracy of $\Delta\mathbf{u}$ and $\Delta\mathbf{z}$ are improved using a vector-valued rational approximation method as described in [12].

We examine a bi-material solid under tension with large deformations; see Fig. 1a. The body is clamped on the left side and loaded by traction $t_x = 50$. The design variables are the Young's moduli

Table 2: Bi-material solid under tension ($J(\boldsymbol{s})(\boldsymbol{u}) = \sigma_{xx}(\boldsymbol{X}_p)$): Accuracy of the different methods (see Table 1). The results are given for different n, the number of iterations used within the reanalysis in methods 1 and 2. The relative errors are given w.r.t. the exact change $\Delta J = \Delta\sigma_{xx}(\boldsymbol{X}_p) = 2.2175$.

Results for $n = 1$:

Method	Approximation of ΔJ	rel. error $[\%]$
1	2.1349	3.7241
2	2.1619	2.5076
3	2.7415	23.628

Results for $n = 2$:

Method	Approximation of ΔJ	rel. error $[\%]$
1	2.1878	1.3397
2	2.2175	0.0018
3	2.7415	23.628

Results for $n = 3$:

Method	Approximation of ΔJ	rel. error $[\%]$
1	2.2179	0.0164
2	2.2176	0.0027
3	2.7415	23.628

Results for $n = 4$:

Method	Approximation of ΔJ	rel. error $[\%]$
1	2.2175	0.0014
2	2.2175	0.0001
3	2.7415	23.628

E_1 and E_2 in the two domains, i.e. $\mathbf{s} = [E_1 \;\; E_2]^T$ and $\Delta\mathbf{s} = [\Delta E_1 \;\; \Delta E_2]^T$. For simplicity, we consider only changes in the second variable E_2 and keep E_1 constant. For the initial design $\mathbf{s}_0$, we chose $E_1 = E_2 = 100$. The Poisson's ratio for both domains is $\nu = 0.3$, and the finite element discretization consists of classical bilinear Q4 elements.

The quantity of interest is the stress component σ_{xx} at point $\boldsymbol{X}_p$, i.e. $J(\boldsymbol{s})(\boldsymbol{u}) = \sigma_{xx}(\boldsymbol{X}_p)$. The dual load case $\mathbf{j}$ (red arrows in Fig. 1d causes an approximation of a unit dislocation at point $\boldsymbol{X}_p$. The primal and dual solutions for the initial design are given in Fig. 1b and Fig. 1e.

In this example, a significant change in Young's moduli is considered, i.e., we investigate a design change $\Delta\mathbf{s} = [0 \; 50]^T$. This yields a significant change in the primal solution; see Fig. 1c. The difference in the dual solution due to the design change is relatively small; see Fig. 1f.

The values of J for the initial and changed designs, as well as the exact change, are given as

$$J(\mathbf{s}_0)(\mathbf{u}_0) = 50.3917, \qquad J(\mathbf{s}_c)(\mathbf{u}_c) = 52.6092, \qquad \Delta J = \Delta\sigma_{xx}(\boldsymbol{X}_p) = 2.2175. \tag{30}$$

Approximations of the changes in the quantity of interest are computed using the three methods given in Table 1. The results are stated in Table 2.

Methods 1 and 2 produce very accurate results despite significant design changes. Method 3, however, provides only a rough approximation for substantial design changes because it is a first-order relation. In this case, method two performs better than method one because the change in the dual solution $\Delta\mathbf{z}$ due to design changes $\Delta\mathbf{s}$ is relatively tiny. As a result, we can obtain a good approximation of $\Delta\mathbf{z}$ with a few iterations using the reanalysis method. On the other hand, the change $\Delta\mathbf{u}$ due to design changes $\Delta\mathbf{s}$ is substantial, requiring more iterations within the reanalysis method to compute $\Delta\mathbf{u}$ with high accuracy.

6 CONCLUSIONS

Reanalysis methods may significantly reduce computational effort in applications involving multiple design modifications, such as structural optimization and reliability and damage analysis. This study presents a goal-oriented reanalysis method for predicting changes in a quantity of interest resulting from structural modifications. This method utilizes changes in the primal and dual solutions.

This paper introduces a method for predicting the change in a quantity of interest ΔJ resulting from structural modifications. The technique calculates changes in J by using the primal or dual solutions changes using the *residual increment approximation* (RIA) method. This reanalysis process is centered around a formulation that involves residual increments of the primal or dual problem. Unlike other reanalysis methods, such as the CA method, which requires evaluating changed stiffness matrices, the RIA method only requires computing and storing residual vectors.

Our proposed methodology is highly adaptable and can be utilized across various design adjustments. The reanalysis process for attaining specific objectives is straightforward. It can be seamlessly integrated into existing finite element programs, as it does not necessitate derivatives concerning the design variables. The numerical example shows the method's ability to produce accurate results despite significant design changes.

In the present paper, we consider only linear problems. Future work will include a discussion of the extension to nonlinear problems.

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