

Effect of the mean stress and the third invariant in fatigue life assuming random loading

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Abstract. This work seeks to present a way to calculate the mean stress under conditions of random loading, as well as to show the effect of the third invariant of the deviatoric stress tensor in the prediction of fatigue life, under low and high number of cycles. Thus, it is proposed to use the so-called “exponential moving average scheme” to determine the mean stress, as well as a modification of the Gao yield criterion, to couple the effect of the mean value and use the third invariant to calculate the equivalent stress. The fatigue life is calculated based on incremental damage proposed by Lemaitre and modified by Malcher & Mamiya. The proposed approach shows that the reduction of the mean stress transient strongly affects the calculation of fatigue life. In addition, the calculation of the equivalent stress, taking into account the third invariant, mainly under shear loading conditions, also has a great influence on the life calculation. Finally, the proposal shows that incremental damage is a valid alternative for calculating fatigue life under random loading conditions.

1 INTRODUCTION

Determining the mean value in a random loading history is an important task and a fundamental data in the correct determination of fatigue life. Through traditional strategies, as the simple weighted average (SWA), the behaviour of the mean stress value can present a large transient until the values felt by the material, see Desmorat et al (2015). This stress transient causes the wrong calculation of the degradation value of the material and therefore the fatigue of a mechanical component in life. Thus, new strategies must be proposed in order to reduce the stress transient, when there is a variation in search of the mean stress levels. In this sense, it is proposed to use the exponential moving average (EMA) to calculate the mean stress under random loading conditions.

2 DEFINITION OF THE MAIN EFFECTS

2.1 – Mean value in random loading.

Regarding x as a variable dependent on the pseudo-time t , and assuming a distribution with constant amplitude, the mean value of x is define based on maximum e minimum values, as:

$$x_{mean} = \frac{(x_{max} + x_{min})}{2} \quad (1)$$

where, x_{mean} represents the mean value for a constant amplitude history, x_{max} is the maximum value and x_{min} is the minimum value. Regarding a random history, the mean value can be defined as the simple weighted average (SWA), as shown in Eq. (2):

$$\tilde{x} = \frac{\int_0^x \bar{x} |dx|}{\int_0^x |dx|} \quad (2)$$

$$\bar{x} = \frac{1}{2}(x_{n+1} + x_n)$$

$$dx = x_{n+1} - x_n$$

where, \tilde{x} represents the mean value of x in a random history, based on the single weighted average approach, x_{n+1} and x_n are the values of x at the pseudo-time t_{n+1} and t_n , respectively. The terms dx and \bar{x} represent the incremental value of x and its mean value, calculated between the pseudo-time t_{n+1} and t_n . Alternatively, in order to reduce the transient during the pseudo-time, the called exponential moving average (EMA), as mathematically represented by Eq. (3), can be written.

$$\tilde{\tilde{x}} = \frac{(1-w) \int_0^{x^m} \bar{x} |dx| + w \int_{x_{m+1}}^{x_{2m}} \bar{x} |dx|}{(1-w) \int_0^{x^m} |dx| + w \int_{x_{m+1}}^{x_{2m}} |dx|} \quad (3)$$

where z is the number of integration intervals, m is the number of increments by integration intervals, h is the number of increments of the total loading and w represents the weight of the exponential moved average. The relation between z , m and h can be observed by Eq. (4).

$$z = \frac{h}{m} \quad (4)$$

Figure 1 presents a schematic representation for the EMA and Box 1 contends the algorithm for its implementation. The idea of the EMA is that the values in t_{n+1} have different weights than the past values, in t_n . In this way, the average values converge more quickly to the current values than in the SWA method. Figure 2 presents the mean value calculated by SWA

and EMA. It is important to observe that the transient until de new mean value, according to EMA is smaller than SWA.

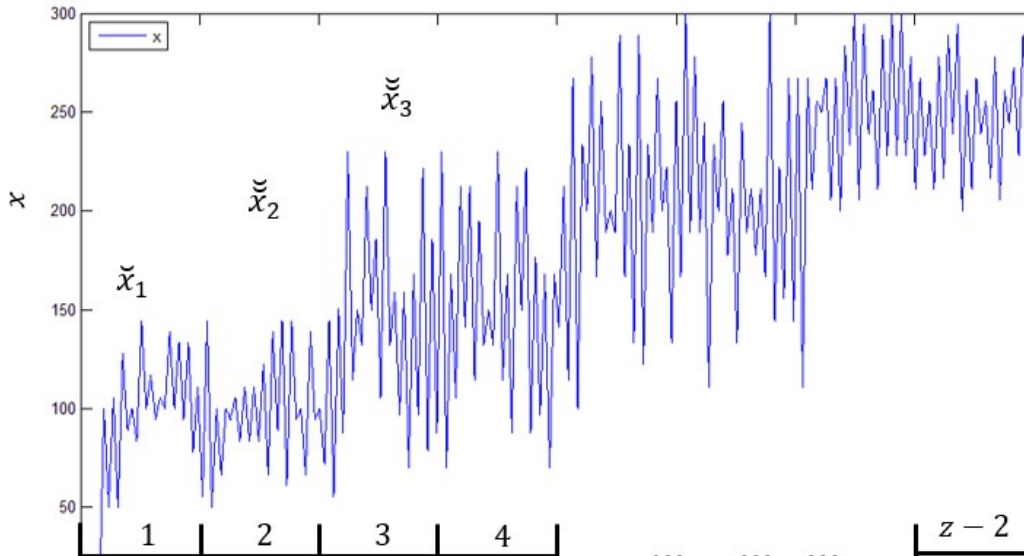


Figure 1. Schematic representation of the EMA approach.

Box 1: Exponential moving average scheme for x

- a) Defined h and z , calculate m : $m = \frac{h}{z}$
- b) Calculate the weight: $w = \frac{2}{z+1}$
- c) SWA for the first integration interval, $i = 1$: EMA=SWA

$$\check{x}_{1(n+1)} = \frac{I_1(n+1)}{x_1^{ac}(n+1)}$$

$$x_{1(n+1)} = x_{1(n)} + \frac{1}{2}(x_{n+1} + x_n)|x_{n+1} + x_n|$$

$$x_1^{ac}(n+1) = x_1^{ac}(n) + |x_{n+1} + x_n|$$

- d) Calculate EMA for the others integration intervals, $2 < i \leq z$:

$$\check{x}_{2(n+1)} = \frac{I_2(n+1)}{x_2^{ac}(n+1)}$$

if $(i + 1) = m$

$$x_2^{ac}(n+1) = (1 - w)x_1^{ac}(n+1) + w|x_{n+1} - x_n|$$

$$I_2(n+1) = (1 - w)I_1(n+1) + w \frac{1}{2}(x_{n+1} + x_n)|x_{n+1} - x_n|$$

else

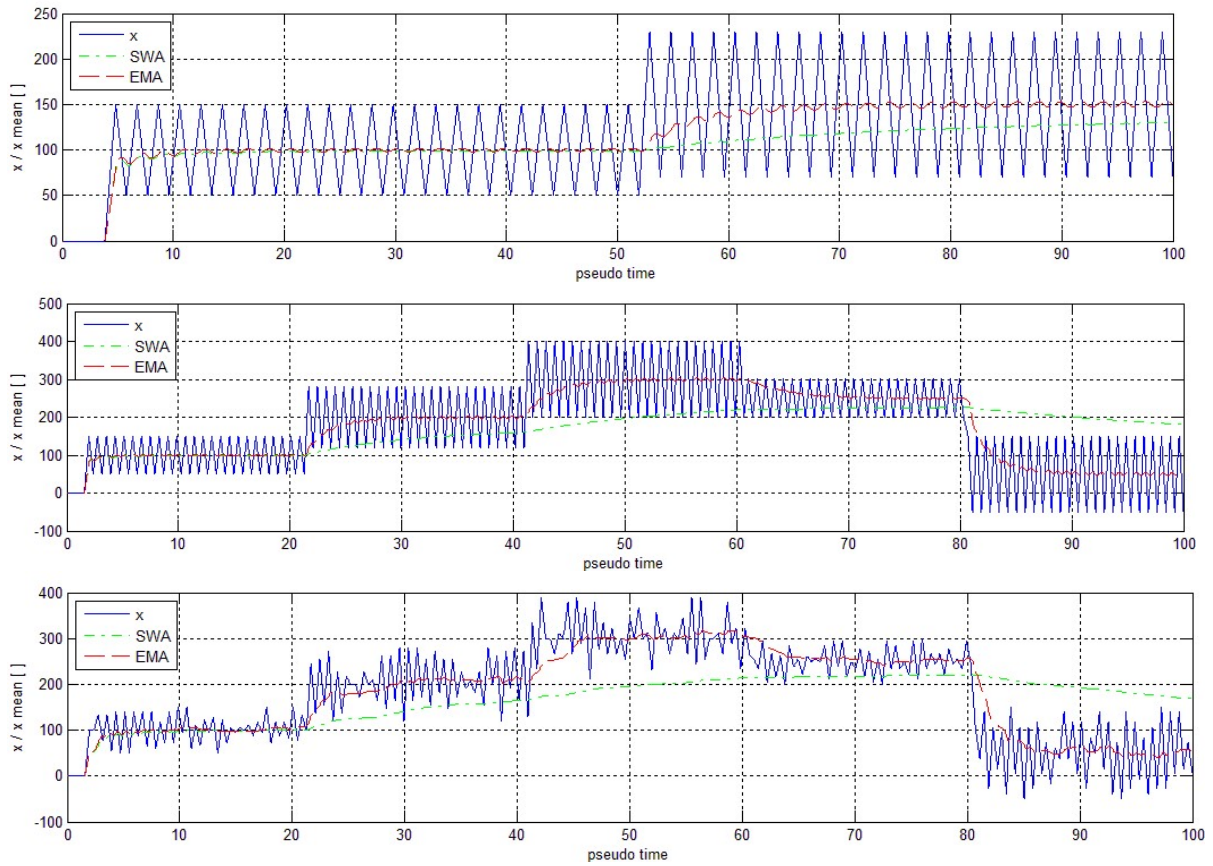


Figure 2. Difference between EMA e SWA approaches and the size of the transient to the new mean value.

3 MEAN STRESS AND YIELD CRITERION

3.1 – Coupling mean stress effect in the yield criterion.

Assuming von Mises yield criterion and the fact that is pressure insensitive, the mean stress effect can be coupled according to Eq. (5).

$$\sqrt{3J_2} + a \tanh \left[\frac{\overset{\sim}{\text{tr}}(\boldsymbol{\sigma})}{3\sigma_f^\infty} \right] = b \quad (5)$$

where J_2 represents the second invariant of the relative stress tensor ($J_2 = \frac{1}{2} \boldsymbol{\eta} : \boldsymbol{\eta}$), $\boldsymbol{\eta} = \boldsymbol{S} - \boldsymbol{\beta}$, which \boldsymbol{S} is the deviatoric stress tensor and $\boldsymbol{\beta}$ is the backstress tensor. The term $\overset{\sim}{\text{tr}}(\boldsymbol{\sigma})$ represents exponential moving average of the trace of $\boldsymbol{\sigma}$ and b can be the fatigue strength or cyclic yield stress of the material. The parameter a is a fitting term and the function $\tanh (*)$ represents the hyperbolic function that try to couple the influence of the mean value by a non-linear function. The Eq. (5) can be rewritten in the normalized form as:

$$\frac{\sqrt{3J_2}}{\sigma_f^\infty} = \frac{b}{\sigma_f^\infty} - \frac{a}{\sigma_f^\infty} \tanh \left[\frac{\text{tr}(\boldsymbol{\sigma})}{3\sigma_f^\infty} \right] \quad (6)$$

where σ_f^∞ represents the fatigue strength of the material. Figure 3 presents the behavior of the Eq. (6) in Haigh diagram for TA6V at room temperature (Desmorat et al, 2015). The parameter b can be calibrated according to tension-compression tests, with $R = -1$, and the parameter a is calibrated regarding axial tests, with $R \neq -1$.

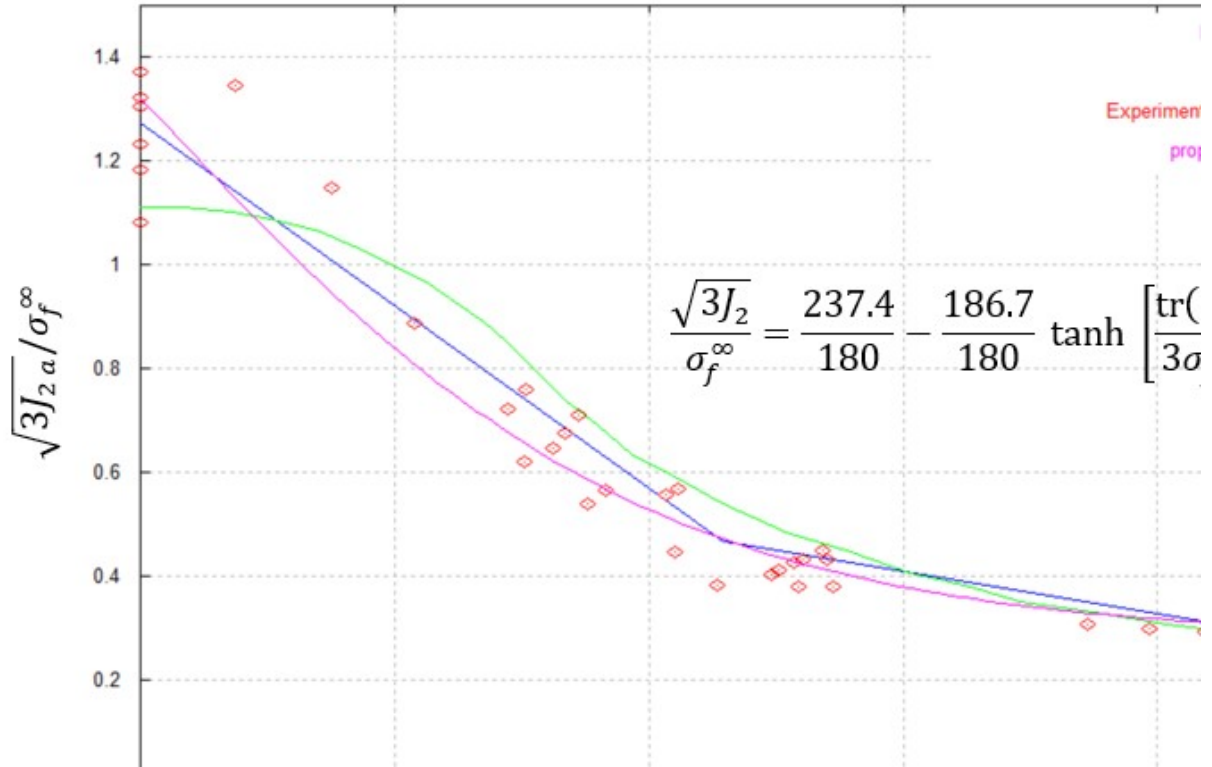


Figure 3. behavior of the Eq. (6) in Haigh diagram for TA6V at room temperature.

3.2 – Define equivalent stress based on J_2 and J_3 .

For many authors, on the definition of the equivalent stress, the magnitude of the stress state can be represented by J_2 and effect of the form of the yield surface is better captured by third invariant of the stress tensor or relative tensor, $J_3 = \det(\boldsymbol{\eta})$. In this sense, if the yield surface of the material is fully regular and symmetric, the effect of J_3 can be negligible and equivalent stress is defined as Mises. However, if the behavior of the material is different in traction and shear, the form of the yield surface can be assumed irregular and the J_3 effect needs to be included on the definition of the equivalent stress. Thus, many authors are trying to redefine a generalized equivalent stress and governed by J_2 and J_3 . Assuming a generalized equivalent stress, Eq. (5) can be rewritten as:

$$\sigma_{eq}(J_2, J_3) + a \tanh \left[\frac{\tilde{\text{tr}}(\boldsymbol{\sigma})}{3\sigma_f^\infty} \right] = b \quad (7)$$

where $\sigma_{eq}(J_2, J_3)$ represents the generalized equivalent stress. In this setting, σ_{eq} can be defined according to Gao (see Gao et al, 2011), Hosford (see Hosford, 1972), Bai et al (see Bai et al, 2007), and many others. Regarding this contribution, the Gao equivalent stress will be assumed, and Eq. (7) is rewritten in the form:

$$d[27J_2^3 + cJ_3]^{\frac{1}{6}} + a \tanh \left[\frac{\tilde{\text{tr}}(\boldsymbol{\sigma})}{3\sigma_f^\infty} \right] = b \quad (8)$$

where the parameter c define the influence of the third invariant on the mechanical behavior of the material and needs to be calibrated according to reversible shear loading tests. The parameter d is defined as:

$$d = \left[\frac{4}{728}c + 1 \right]^{-\frac{1}{6}} \quad (9)$$

The behavior of Eq. (8) can be observed as: a) If a is equal to zero, the mean stress effect is negligible; b) If c is equal to zero, the parameter $d = 1$ and the third invariant effect is lost; c) If a and c are equal to zero, both effects are uncoupled and the model recover the von Mises behavior; d) If a is equal to zero and c assumes large values, the model recover the Tresca behavior; e) Regarding shear loading, the trace of $\boldsymbol{\sigma}$ is equal to zero and the mean stress does not affect the behavior of the material, which is observed in the literature.

4 INCREMENTAL DAMAGE IN FATIGUE LIFE

4.1 – Incremental damage and $S(\Gamma)$.

Assuming the incremental damage to estimate fatigue life, in this contribution, the fracture indicator proposed by Vaz (see Vaz, 1998) is modified including a generalized equivalent stress, a denominator of function dependent on the stress state and an application in random loading. In this setting, the original fracture indicator proposed by Vaz can be mathematically represented, as:

$$I = \int_0^{\bar{\varepsilon}_f^p} \left[\frac{1}{S_0} \left(\frac{\sigma_{eq}^2}{6G} + \frac{p^2}{2K} \right) \right]^s d\bar{\varepsilon}^p \quad (10)$$

where $\bar{\varepsilon}_f^p$ represents the accumulated plastic strain at fracture, σ_{eq} is the Mises equivalent stress, $p = \frac{\text{tr}(\boldsymbol{\sigma})}{3}$ is the hydrostatic pressure, G and K are the shear and volumetric modulus,

$d\bar{\varepsilon}^p$ is the incremental equivalent plastic strain and S_0 and s are, respectively, the denominator and exponent of damage. Regarding the proposition of Malcher and Mamiya (see Malcher and Mamiya, 2014) that S_0 cannot be a constant, but a function of the stress triaxiality and third invariant, as:

$$S(\Gamma, \xi) = \frac{S_{\frac{1}{3}}}{3|\Gamma| + \frac{S_{\frac{1}{3}}}{S_0}(1 - \xi^2)} \quad (11)$$

where S represents a function denominator of damage, $S_{\frac{1}{3}}$ and S_0 are the denominator of damage calibrated in axial and shear cyclic loading with $R = -1$, $\Gamma = \frac{p}{\sigma_{eq}}$ is the stress triaxiality and $\xi = \frac{\frac{27}{2}J_3}{(3J_2)^{\frac{3}{2}}}$ is the normalized third invariant. Assuming now a plane stress and the relation between Γ and ξ proposed by Wierzbicki and Xue (see Wierzbicki and Xue, 2005), $\xi = -\frac{27}{2}\Gamma(\Gamma^2 - \frac{1}{3})$, Eq. (11) can be rewritten as a function only the stress triaxiality.

$$S(\Gamma) = \frac{S_{\frac{1}{3}}}{3|\Gamma| + \frac{S_{\frac{1}{3}}}{S_0}\left[1 - \frac{729}{4}\Gamma^2(\Gamma^2 - \frac{1}{3})^2\right]} \quad (12)$$

Regarding Eq. (12), the concept of Gao generalized equivalent stress and mean stress calculated by EMA, Eq. (10) can be rewritten for a random loading as:

$$I = \int_0^{\bar{\varepsilon}_f^p} \left\{ \frac{1}{S(\Gamma)} \left[\frac{d^2(27J_2^3 + cJ_3)^{\frac{2}{6}}}{6G} + \frac{\left(\frac{\check{\text{tr}}(\sigma)}{3}\right)^2}{2K} \right] \right\}^s d\bar{\varepsilon}^p \quad (13)$$

where J_2 and J_3 are calculated at each pseudo-time t_{n+1} . The stress triaxiality is determined as:

$$\Gamma = \frac{\frac{\check{\text{tr}}(\sigma)}{3}}{d(27J_2^3 + cJ_3)^{\frac{1}{6}}} \quad (14)$$

4 CONCLUSIONS

The SWA method significantly reduces the transient in calculating the mean mechanical stress, compared to the EMA method. What can cause a significant change in the calculation of fatigue life under random loading conditions. It is also found that through a single

hyperbolic equation, it is possible to describe the relationship between equivalent stress amplitude and mean stress.

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