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PLASTIC ANALYSIS OF TRUSS STRUCTURES UNDER RANDOM STRENGTH BY CHANCE CONSTRAINED PROGRAMMING

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Summary. This paper shows how probabilistic limit analysis of statically indeterminate truss structures can be done with the same simplicity as the deterministic limit analysis.

1 INTRODUCTION

As alternative to plastic reliability analysis [1] the authors have developed FEM based limit and shakedown analysis of structural problems with uncertain data by stochastic optimization [2,3]. They have developed chance constrained programming with individual chance constraints for normally and lognormally distributed strength and loading to calculate limit and shakedown loads for prescribed reliability levels.

Plastic limit analysis of structures can be formulated as a linear program (LP) [4,5]. It is the mainly used plastic design method in civil engineering practice and teaching in the analysis and design of statically indeterminate truss and frame structures. Therefore the chance constrained program with individual constraints is formulated here for trusses to demonstrate the application of the concept to typical truss girder problems. Limit analysis of frame structures is discussed in [6].

2 STATIC APPROACH FOR LIMIT ANALYSIS OF TRUSS UNDER RANDOM STRENGTH

For the static approach, we are looking for the maximum safe load for a statically admissible stress field:

$$\alpha_{\lim} = \max \alpha$$
$$\text{s.t.:} \begin{cases} \mathbf{BS} = \alpha \mathbf{f} \\ -\mathbf{S}_0 \le \mathbf{S} \le \mathbf{S}_0 \end{cases} \tag{1}$$

In (1) the equality constraint are the equilibrium equations.
$\mathbf{B}$ is the equilibrium matrix; vector $\mathbf{S}$ contains internal the forces of bars

$$\mathbf{S} = \left[S_1, ..., S_n \right]^T$$

$$\mathbf{S}_0 = \left[A_1\sigma_0, A_2\sigma_0, A_3\sigma_0, ..., A_n\sigma_0 \right]^T$$

$\mathbf{f}$ is the vector of members forces acting on the joints of the truss

The problem (1) can be rewritten as

$$\alpha_{\lim} = \max \alpha$$
$$\text{s.t.:} \begin{cases} \mathbf{BS} = \alpha \mathbf{f} \\ |\mathbf{S}| - \mathbf{S}_0 \le 0 \end{cases} \tag{2}$$

The maximum problems (1) and (2) are linear programs (LP).

Consider the situation that the yield stress of the material is not given but must be modelled through random variables $r = r(\omega)$ in a certain probability space. Under uncertainty, the inequalities of are not always satisfied, the probability of the i^{th} yield condition being satisfied is required to be greater than some reliability level ψ_i. Problem (2) becomes an individually chance constrained programming problem:

$$\alpha_{\lim} = \max \alpha$$
$$\text{s.t.:} \begin{cases} \mathbf{BS} = \alpha \mathbf{f} \\ \text{Prob}\left(|\mathbf{S}| - \mathbf{S}_0\,(\omega) \le 0 \right) \ge \psi \end{cases} \tag{3}$$

Let us consider the individual chance constraint:

$$\text{Prob}\left[|\mathbf{S}_i| - S_{0i}(\omega) \le 0 \right] \ge \psi_i \tag{4}$$

2.1 The normally distributed strength

We assume that the fully plastic internal forces $S_{0i}(\omega)$ of the material follows a Gaussian distribution $\mathcal{N}(\mu_i, \sigma_i)$ with mean value μ_i and standard deviation σ_i. Let us transform to a standard normal distribution. The yield condition can be written as $\frac{|S_i| - \mu_i}{\sigma_i} \le \frac{S_{0i}(\omega) - \mu_i}{\sigma_i}$ and we have:

$$\text{Prob}\left[|S_i| \le S_{0i}(\omega)\right] = \text{Prob}\left[\frac{|S_i| - \mu_i}{\sigma_i} \le \frac{S_{0i}(\omega) - \mu_i}{\sigma_i}\right]$$

Using the property of the cumulative distribution function (c.d.f.) of the standard normal distribution $\Phi(-x) = 1 - \Phi(x)$, we can write as follows:

$$\text{Prob}\left[\frac{|S_i| - \mu_i}{\sigma_i} \le \frac{S_{0i}(\omega) - \mu_i}{\sigma_i}\right] = 1 - \Phi\left(\frac{|S_i| - \mu_i}{\sigma_i}\right) = \Phi\left(\frac{\mu_i - |S_i|}{\sigma_i}\right)$$

Now the probabilistic condition (4) is replaced by

$$\Phi\left[\frac{\mu_i - |S_i|}{\sigma_i}\right] \ge \psi_i . \tag{5}$$

Introducing a new variable $\kappa_i = \Phi^{-1}(\psi_i)$ so that $\psi_i = \Phi(\kappa_i)$, inequality (5) becomes:

$$\Phi\left[\frac{\mu_i - f_i}{\sigma_i}\right] \ge \Phi(\kappa_i) . \tag{6}$$

Because Φ is monotonic it holds

$$\kappa_i \le \frac{\mu_i - |S_i|}{\sigma_i} \quad \text{or} \quad |S_i| \le \mu_i - \kappa_i \sigma_i . \tag{7}$$

Finally we get an equivalent deterministic formulation of the static approach:

$$\alpha_{\lim} = \max \alpha$$

$$\text{s.t.:} \begin{cases} \mathbf{BS} = \alpha \mathbf{f} \\ -\mu_i + \kappa_i \sigma_i \le S_i \le \mu_i - \kappa_i \sigma_i & \qquad i = 1 ... N_b \end{cases} \tag{8}$$

N_b is the number of bar of truss.

2.2 The lognormally distributed strength

If the yield limit of truss bars $S_{0i}(\omega)$ is distributed lognormally with parameters μ_i and σ_i then $\ln\left[S_{0i}(\omega)\right]$ is distributed normally with mean μ_i and standard deviation σ_i, in short $\ln S_{0i}(\omega) \sim \mathcal{N}(\mu_i, \sigma_i^2)$.

The probabilistic constraint (4) can be rewritten with the complementary cumulative distribution function:

$$1 - \text{Prob}\left[S_{0i}(\omega) \le |S_i|\right] \ge \psi_i \tag{9}$$

Similar to the case of normally distributed strength, we would like to find an equivalent deterministic of problem (3). Let us make some transformations:

$$\begin{aligned}\text{Prob}\left[S_{0i}(\omega) \le |S_i|\right] &= \text{Prob}\left\{\ln\left[S_{0i}(\omega)\right] \le \ln\left[|S_i|\right]\right\} \\ &= \text{Prob}\left[\frac{\ln\left[S_{0i}(\omega)\right]-\mu_i}{\sigma_i} \le \ln\left(\frac{\ln\left[|S_i|\right]-\mu_i}{\sigma_i}\right)\right] \\ &= \Phi\left[\frac{\ln\left[|S_i|\right]-\mu_i}{\sigma_i}\right]\end{aligned} \tag{10}$$

By the same argument of the case of normally distributed strength of bars, we can prove that inequality (9) is equivalent with

$$\kappa_i \le \frac{\mu_i - \ln\left(|S_i|\right)}{\sigma_i} \tag{11}$$

From (11) we have:

$$|S_i| \le e^{\mu_i - \kappa_i \sigma_i} \tag{12}$$

Finally we get an equivalent static deterministic program for lognormally distributed strength:

$$\begin{aligned}&\alpha_{\text{lim}} = \max \alpha \\ &\text{s.t.:} \begin{cases}\mathbf{BS} = \alpha \mathbf{f} \\ -e^{\mu_i - \kappa_i \sigma_i} \le S_i \le e^{\mu_i - \kappa_i \sigma_i} & i = 1,\dots,n\end{cases}\end{aligned} \tag{13}$$

Note that in (13) μ, κ are parameters of lognormal distribution and they relate with S_0 by following equations:

$$\mu = \ln\left(\frac{E^2\left[s_0\right]}{\sqrt{Var\left(s_0\right) + E^2\left[s_0\right]}}\right), \qquad \sigma = \sqrt{\ln\left(\frac{Var\left(s_0\right)}{E^2\left[s_0\right]} + 1\right)}$$

3 EXAMPLES

3.1 Example 1

Consider a truss structure subjected to a horizontal force F with the topology shown in Figure 1. All bars of the truss have the same cross section area $A_1 = A_2 = A_3 = A_4 = A_5 = A$. They are made from a material with the yield stress $\hat{\sigma}_0$, which has the same value in tension and compression. Thus for all member forces $-S_0 \leq S_i \leq S_0$ with $S_0 = \sigma_0 A$ for $i = 1,...,5$. We compute the limit load $F_{\lim}$ in the following situations:

- The yield stress is deterministic
- The yield stress is a random variable, which is distributed normally or lognormally with the mean value $E\left[\hat{\sigma}_0(\omega)\right]$ and the standard deviation $\sigma = 0.1E\left[\hat{\sigma}_0(\omega)\right]$

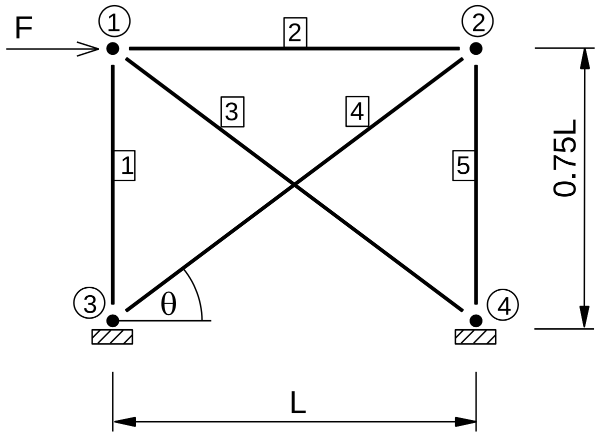

Figure 1: Truss girder loaded by a horizontal load F [5]

For the deterministic analysis, we use the LP (1)

Equilibrium equations are written for joints 1 and 2:

Joint 1:

$$\sum X = S_2 + S_3 \cos\theta + F = 0$$
$$\sum Y = S_1 + S_3 \sin\theta = 0$$

Joint 2:

$$\sum X = S_2 + S_4 \cos\theta = 0$$
$$\sum Y = {}_1 S_4 \sin\theta + S_5 = 0 \qquad (14)$$

$$F_{\lim} = \max F$$
$$\text{s.t.} \begin{cases} S_2 + S_3 \cos\theta + F = 0 \\ S_1 + S_3 \sin\theta = 0 \\ S_2 + S_4 \cos\theta = 0 \\ S_4 \sin\theta + S_5 = 0 \\ -S_0 \le S_1 \le S_0 \\ -S_0 \le S_2 \le S_0 \\ -S_0 \le S_3 \le S_0 \\ -S_0 \le S_4 \le S_0 \\ -S_0 \le S_5 \le S_0 \end{cases} \tag{15}$$

Problem (15) is equivalent to problem (1) with

$$\mathbf{B} = \begin{bmatrix} 0 & 1 & \cos\theta & 0 & 0 \\ 1 & 0 & \sin\theta & 0 & 0 \\ 0 & 1 & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & \sin\theta & 1 \end{bmatrix}, \qquad \mathbf{S} = \left[S_1, S_2, S_3, S_4, S_5\right]^T, \qquad \mathbf{f} = \left[-1\ 0\ 0\ 0\right]^T$$

Problem (15) is a LP, which can be solved by the *linprog* or the *fmincon* function of the optimization toolbox of MATLAB. The results are listed in Table 1, they are the same as the results in [5].

Table 1: Results for the case of the determinsitic problem

Unknowns	Present	Jirásek, Bažant [5]
S_1	$0.6S_0$	$0.6S_0$
S_2	$-0.8S_0$	$-0.8S_0$
S_3	$-S_0$	$-S_0$
S_4	S_0	S_0
S_5	$-0.6S_0$	$-0.6S_0$
Limit load factor α	1.6	1.6

- For the normal distribution of the yield stress, the reliability level $\psi = 0.9999$ gives us $\kappa = 3.719$. For this example the limit load factor is obtained by solving LP (8) with $\mu_i = S_0$, $\sigma_i = 0.1S_0$, $\kappa_i = 3.719$
- For the lognormal distribution of yield stress, we solve LP (13) with the parameters

$$\mu = \ln\left(\frac{E^2\left[s_0\right]}{\sqrt{Var\left(s_0\right)+E^2\left[s_0\right]}}\right) = -0.004975 \qquad \sigma = \sqrt{\ln\left(\frac{Var\left(s_0\right)}{E^2\left[s_0\right]}+1\right)} = 0.09975$$

The results for the case of random strength are shown in Table 2. The results for different reliability levels are shown in Table3 and Figure 2.

Table 2: Results for the case of random strength and reliability level $\psi = 0.9999$

Unknowns	Normal	Lognormal
S_1	$0.377S_0$	$0.412S_0$
S_2	$-0.502S_0$	$-0.549S_0$
S_3	$-0.628S_0$	$-0.686S_0$
S_4	$0.628S_0$	$0.686S_0$
S_5	$-0.377S_0$	$-0.412S_0$
Limit load factor α	1.0049≈1	1.0986≈1.1

Table 3: Limit load factors corresponding to some reliability levels and failure probabilities

Reliability level ψ	Failure Probability P_f	Normal	Lognormal
0.9	10^{-1}	1.3949	1.4010
0.99	10^{-2}	1.2278	1.2624
0.999	10^{-3}	1.1056	1.1679
0.9999	10^{-4}	1.0049	1.0986
0.99999	10^{-5}	0.9176	1.0404

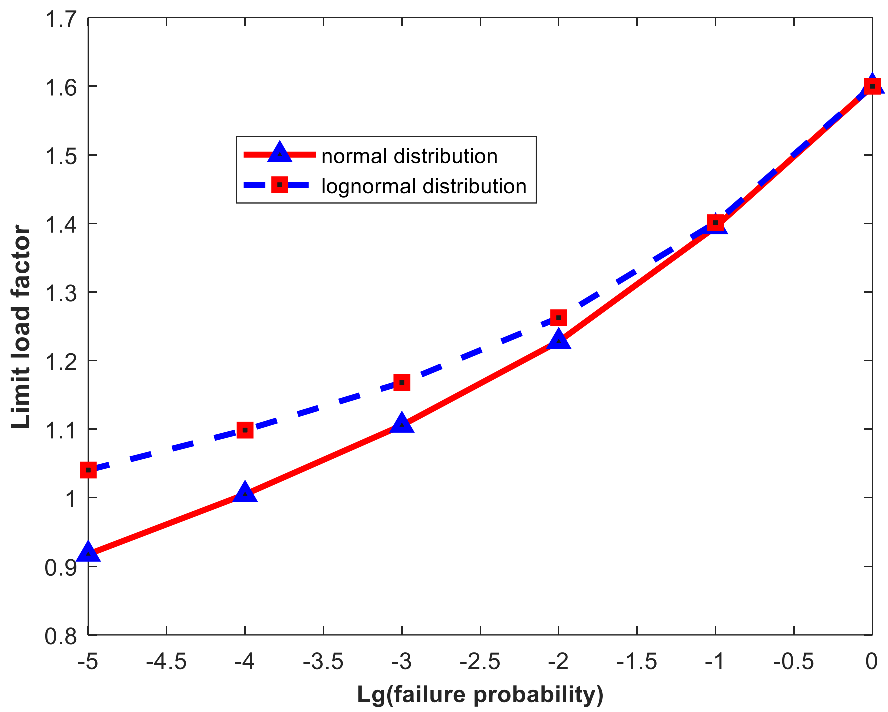

Figure 2: Relation between the failure probability and limit load factors

3.2 Example 2

In the second example, we consider a 10-bar truss with the topology shown in Figure 3. The force of full plasticity of the 5 bars $S_1,...,S_5$ in the botom part is $2S_0$, the force of the full plasticity of the 5 bars $S_6,...,S_{10}$ in the top part of he structure is S_0. The forces of full plasticity are the same in compression and in tension.

We calculate the maximum of applied horizontal loads F in the following situations :

- The forces $2S_0$, S_0 of full plasticity of the bars are deterministic
- The forces of full plasticity of bars are normally distributed with mean μ_i and standard deviation $\sigma_i = 0.1\mu_i$, respectively ($\mu_i = 2S_0$ for $i = 1,...,5$ and $\mu_i = S_0$ for $i = 6,...,10$) The reliability level is assumed $\psi = 0.9999$ so that $\kappa = 3.719$.

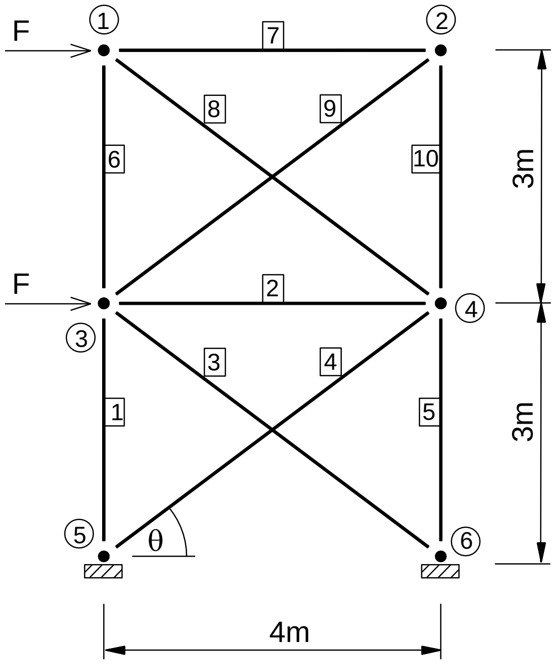

Figure 3: Truss girder loaded by horizontal loads F [7]

For the deteministic problem, LP (1) is used with

$$\mathbf{B} = \begin{bmatrix} 0 & 1 & 0.8 & 0 & 0 & 0 & 0 & 0 & 0.8 & 0 \\ -1 & 0 & -0.6 & 0 & 0 & 1 & 0 & 0 & 0.6 & 0 \\ 0 & 1 & 0 & 0.8 & 0 & 0 & 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & -0.6 & -1 & 0 & 0 & 0.6 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0.6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0.8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.6 & 1 \end{bmatrix}$$

$$\mathbf{S} = \left[S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8, S_9, S_{10}\right]^T$$

$$\mathbf{f} = \left[-1\ 0\ 0\ 0\ -1\ 0\ 0\ 0\right]^T$$

The mean values and standard deviations of normally distributed full plasticity forces are:

$$\mu_i = 2S_0 \qquad \sigma_i = 0.2S_0 \qquad i = 1,...,5$$

$$\mu_i = S_0 \qquad \sigma_i = 0.1S_0 \qquad i = 6,...,10$$

For lognormally distributed full plasticity forces, μ_i, σ_i are the parameters of the respective lognormal distribution.

In case of random full plasticity forces of each bar, we use LP (8) or (13). The results for different distributions are listed in Table 4.

Table 4: Limit load factors and internal forces of bars for $\psi = 0.9999$ in comparison

Unknowns	Deterministic	Deterministic	Normal	Lognormal
	Petrović et al. [7]		Present	
S_1	$2.0S_0$	$2.0S_0$	$1.256S_0$	$1.373\,S_0$
S_2	$-0.533S_0$	$-0.533S_0$	$-0.335S_0$	$-0.366\,S_0$
S_3	$-1.667S_0$	$-1.667S_0$	$-1.047S_0$	$-1.144\,S_0$
S_4	$1.667S_0$	$1.667S_0$	$1.047S_0$	$1.144\,S_0$
S_5	$-2.0S_0$	$-2.0S_0$	$-1.256S_0$	$-1.373\,S_0$
S_6	$0.6S_0$	$0.6S_0$	$0.3768S_0$	$0.412\,S_0$
S_7	$-0.533S_0$	$-0.533S_0$	$-0.335S_0$	$-0.366\,S_0$
S_8	$-S_0$	$-S_0$	$-0.628S_0$	$-0.686\,S_0$
S_9	$0.667S_0$	$0.667S_0$	$0.418S_0$	$0.458\,S_0$
S_{10}	$-0.4S_0$	$-0.4S_0$	$-0.251S_0$	$-0.275\,S_0$
$F_{\lim}$	$1.333S_0$	$1.333S_0$	$0.837S_0$	$0.916\,S_0$

4 CONCLUSIONS

Full Probabilistic limit analysis can be made with the deterministic equivalent of the chance constraints for normally or lognormally distributed strength data. Then the analysis is basically the same as a deterministic limit analysis. The limit loads are obtained for any target reliability level, if the mean value and standard deviation of strength are available. This is equivalent to material partial safety factors that depend on the target reliability level. The extension for load partial safety factors is discussed in [3]. Any statically indeterminate truss structure can be handled in the demonstrated way.

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