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A COMPACT SIXTH ORDER FINITE DIFFERENCE SCHEME FOR THE 3D POISSON EQUATION

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1 INTRODUCTION

We consider the three-dimensional Poisson's equation in a rectangle $\Omega \subset \mathbb{R}^3$ with appropriate boundary conditions on $\partial\Omega$

$$\Delta\phi = f, \text{ in } \Omega \tag{1}$$

Employing the conventional *7-point* discrete Laplacian $L_7^h\phi_{\mathbf{i}} = \sum_{d=0}^{2} \phi_{\mathbf{i}-\mathbf{e}^d} - 2\phi_{\mathbf{i}} + \phi_{\mathbf{i}+\mathbf{e}^d}$, where $\mathbf{e}^d$ is the unit vector in the d direction, results in a second order accurate scheme. The Laplace operator can be discretized in a compact fashion in order to obtain high order accuracy by utilizing compact or so called *Mehrstellen* discretizations [1], that is stencils of small radius measured as the maximum distance in points from the stencil's evaluation point. In the present study we consider the L_{19}^h and L_{27}^h Mehrstellen discretizations of the Laplacian [2, 3]. The coefficients for both stencils are given by $a_{\boldsymbol{j}} = \frac{1}{h^2} b_{|\boldsymbol{j}|}$, where $|\boldsymbol{j}|$ is the number of non-zero components of $\boldsymbol{j}$ and b_k are defined as

$$\begin{aligned}
&b_0 = -4, \qquad b_1 = \frac{1}{3}, \qquad b_2 = \frac{1}{6}, \qquad b_3 = 0, \quad \textit{19-point stencil}\\
&b_0 = -\frac{64}{15}, \qquad b_1 = \frac{7}{15}, \qquad b_2 = \frac{1}{10}, \qquad b_3 = \frac{1}{30}, \quad \textit{27-point stencil}
\end{aligned}$$

The associated truncation errors $\tau_k^h = (L_k^h - \Delta)\phi,\ k = 19,\ 27$ are given by

$$\tau_{19}^h(\boldsymbol{x}) = \frac{h^2}{12}\Delta^2\phi(\boldsymbol{x}) + \frac{h^4}{360}\left(\Delta^3 + 2\left(\frac{\partial^4\Delta}{\partial x^2\partial y^2} + \frac{\partial^4\Delta}{\partial y^2\partial z^2} + \frac{\partial^4\Delta}{\partial z^2\partial x^2}\right) - 12\frac{\partial^6}{\partial x^2\partial y^2\partial z^2}\right)\phi(\boldsymbol{x}) + O(h^6) \tag{2}$$

and

$$
\begin{aligned}
\tau_{27}^h(\boldsymbol{x}) \;=\;& \frac{h^2}{12}\Delta^2\phi(\boldsymbol{x}) + \frac{h^4}{360}\left(\Delta^3 + 2\left(\frac{\partial^4\Delta}{\partial x^2\partial y^2} + \frac{\partial^4\Delta}{\partial y^2\partial z^2} + \frac{\partial^4\Delta}{\partial z^2\partial x^2}\right)\right)\phi(\boldsymbol{x}) + \\
& \frac{h^6}{60480}\left(3\Delta^4 + 16\left(\frac{\partial^4\Delta^2}{\partial x^2\partial y^2} + \frac{\partial^4\Delta^2}{\partial y^2\partial z^2} + \frac{\partial^4\Delta^2}{\partial z^2\partial x^2}\right) + \right. \\
& \left. 52\frac{\partial^6\Delta}{\partial x^2\partial y^2\partial z^2} + 20\left(\frac{\partial^8}{\partial x^4\partial y^4} + \frac{\partial^8}{\partial y^4\partial z^4} + \frac{\partial^8}{\partial z^4\partial x^4}\right)\right)\phi(\boldsymbol{x}) + O(h^8) \qquad (3)
\end{aligned}
$$

and as is evident they are fourth and sixth order accurate for harmonic functions, respectively, and only second order accurate for a general non-vanishing f. It would be desirable to retain the high order of accuracy of both stencils when $f \not\equiv 0$. Indeed, as is well known [1, 3], replacing f with the modified right hand side

$$
f^{*,h} = f^h + \frac{h^2}{12}\mathcal{L}^h f^h \qquad (4)
$$

where $\mathcal{L}^h$ is any second order accurate approximation of the Laplacian, results in a fourth order scheme for nonzero f. The aim of the present study is the derivation of a Mehrstellen correction of f so that the associated numerical solution is sixth order accurate in the case of L_{27}^h. This has been accomplished in [3] for the special case of analytically known fourth order derivatives of f. In particular, the following compact scheme is discussed there:

$$
\begin{aligned}
\text{Compute } \phi^h \text{ using } L_{27}^h\phi^h \;&=\; f^* \text{ where} \\
f_{ijk}^* \;&=\; f_{ijk} + \frac{h^2}{12}\Delta f_{ijk} + \frac{h^4}{360}Bf_{ijk} \\
&+\; \frac{h^4}{180}\left(\frac{\partial^4 f}{\partial x^2\partial y^2} + \frac{\partial^4 f}{\partial y^2\partial z^2} + \frac{\partial^4 f}{\partial x^2\partial z^2}\right)
\end{aligned}
$$

If the fourth order derivatives of f are known explicitly this scheme is sixth order accurate as follows from the form of the Laplace operator truncation error in (3). The work in [3] is extended here to handle the general case where f is given as gridded data, that is, f is known at grid points and no information regarding its derivatives is available. A sixth order scheme based on evaluation of f at non-stencil points has appeared in [4, 5], however, in this work we are considering values of the right hand side at grid points only.

2 SIXTH ORDER MEHRSTELLEN CORRECTION FOR L_{27}^h

In this section we present three forms of a Mehrstellen correction for the L_{27}^h Mehrstellen Laplacian that result in an $O(h^6)$ truncation error. The truncation error for L_{27}^h is given by (3) and can be written as [3, 5]

$$
\tau^h(\boldsymbol{x}) = (L_{27}^h - \Delta)\phi(\boldsymbol{x}) = \frac{h^2}{12}\Delta^2\phi(\boldsymbol{x}) + \frac{h^4}{360}\Delta^3\phi(\boldsymbol{x}) + \frac{h^4}{180}D(\Delta\phi)(\boldsymbol{x}) + O(h^6)
$$

where $D = \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^2 \partial z^2} + \frac{\partial^4}{\partial z^2 \partial x^2}$. If we denote the second order central discrete derivatives by D^h_{xx}, D^h_{yy}, D^h_{zz} and define $D^h = D^h_{xx} D^h_{yy} + D^h_{yy} D^h_{zz} + D^h_{zz} D^h_{xx}$ we obtain a second order approximation to the cross derivatives operator D. Moreover, we note that the discrete biharmonic operator $B^h = L^h_k L^h_l$, where *k,l=7,19 or 27* is also *2nd* order accurate. Turning our attention to the discrete Poisson problem

$$L^h_{27} \phi^h = f^h$$

we note that the associated truncation error is given by

$$\tau^h(\boldsymbol{x}) = L^h_{27}(\phi^h - \phi) = -\frac{h^2}{12}\Delta f(\boldsymbol{x}) - \frac{h^4}{360}\Delta^2 f(\boldsymbol{x}) - \frac{h^4}{180} D f(\boldsymbol{x}) + O(h^6)$$

In the following, we consider suitable modifications of the right hand side that lead to a sixth order truncation error for ϕ^h. Again we emphasize that f is given as a set of values at mesh points only.

Mehrstellen Correction of type A

If we define the modified right hand side

$$f^{*,h} = f^h + \frac{h^2}{12} L^h_{19} f^h - \frac{h^4}{240} B^h f^h + \frac{h^4}{180} D^h f^h \tag{5}$$

and solve $L^h_{27} \phi^{*,h} = f^{*,h}$, then the truncation error $\tau^{*,h} = L^h_{27}(\phi^{*,h} - \phi)$ of the modified problem is sixth order accurate. Indeed,

$$\begin{aligned}
\tau^{*,h} \quad &= \tau + \frac{h^2}{12} L^h_{19} f^h - \frac{h^4}{144} B^h f^h + \frac{h^4}{360} B^h f^h + \frac{h^4}{180} D^h f^h \\
&= -\frac{h^2}{12}\Delta f + \frac{h^2}{12} L^h_{19} f^h - \frac{h^4}{144} B^h f^h + \frac{h^4}{360}(B^h - \Delta^2) f + \frac{h^4}{180}(D^h - D) f + O(h^6) \\
&= \frac{h^2}{12}\left[(L^h_{19} - \Delta) f - \frac{h^2}{12} B^h f^h\right] + O(h^6) \\
&= \frac{h^2}{12}\left[\frac{h^2}{12}\Delta^2 f - \frac{h^2}{12} B^h f^h\right] + O(h^6) = O(h^6)
\end{aligned}$$

It is noted that L^h_{27} may also be used instead of L^h_{19} in the second term of $f^{*,h}$.

Mehrstellen Correction of type B

An alternative form of the Mehrstellen correction is given by

$$f^{*,h} = f^h - \frac{h^2}{30} L^h_7 f^h + \frac{7h^2}{60} L^h_{19} f^h - \frac{h^4}{240} B^h f^h \tag{6}$$

The associated truncation error $\tau^{*,h}$ is again of sixth order. The proof is performed in a similar way making use of the estimate

$$\frac{h^4}{180}Df = \frac{h^2}{30}\Delta f + \frac{h^4}{360}\Delta^2 f - \frac{h^2}{30}L_7^h f + O(h^6)$$

which follows from the form of the truncation error of the *7-point* discrete Laplacian

$$(L_7^h - \Delta)f = \frac{h^2}{12}\left(\frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4} + \frac{\partial^4}{\partial z^4}\right) f + O(h^4)$$

and the identity $\Delta^2 f = \left(\frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4} + \frac{\partial^4}{\partial z^4}\right) f + 2Df$. Again, the discrete operator L_{27}^h may be employed in the third term of (6). It is noted that form **B** for the modified right hand side has a clear advantage over form **A**, since fewer operations per point are needed for the computation of term $\frac{h^2}{30}L_7^h f^h$ compared to term $\frac{h^4}{180}D^h f^h$.

Mehrstellen Correction of type C
With infinite domain boundary conditions, that is when the potential satisfies

$$\phi(\mathbf{x}) = -\frac{1}{4\pi\|\mathbf{x}\|}\int_{\mathbb{R}^3} f(\mathbf{y})d\mathbf{y} + o\left(\frac{1}{\|\mathbf{x}\|}\right), \quad \|\mathbf{x}\| \to +\infty \tag{7}$$

we can first solve with a simpler right hand side and then post-process the resulting solution to obtain sixth order approximation and further reduce the operation count of Mehrstellen Correction **B**. Specifically, we may employ the following scheme:

Solve $L_{27}^h \phi^{*,h} = f^{*,h} \equiv f^h - \frac{h^2}{30}L_7^h f^h$
Compute $\tilde{\phi}^h := \phi^{*,h} + \frac{7h^2}{60} f^h - \frac{h^4}{240}L_7^h f^h$

for which we can verify as before that it is sixth order accurate. Furthermore, we only need values of f^h up to and including the boundary of Ω to apply this Mehrstellen correction, in contrast to forms **A** and **B** that require f^h values at one additional layer of points beyond.

3 APPLICATION TO THE METHOD OF LOCAL CORRECTIONS

In the following, the sixth order Mehrstellen scheme is coupled with the Method of Local Corrections (MLC). The MLC method is a parallel, non-iterative, multilevel method with a minimal amount of communication for computing volume potentials. We describe here the salient points of the simplest two-level MLC method. Further details can be found in [6].

We consider Poisson's equation with free-space boundary conditions given by (7). It is assumed that the support of the charge f is contained in a cube Ω that is further subdivided into a union of disjoint cubic patches $\Omega_{R,\mathbf{i}}$ of radius R. We associate with Ω a fine grid Ω^h and a coarse one Ω^H with mesh spacing h and H, respectively. The grid Ω^h is clearly the union of the grids $\Omega^h_{R,\mathbf{i}}$ associated with each patch $\Omega_{R,\mathbf{i}}$. The restriction of f^h on each subgrid $\Omega^h_{R,\mathbf{i}}$ is denoted by $f^{\mathbf{i},h}$. Also we denote by $\Omega^h_{R,\mathbf{i},\beta}$ the extended patches with radius βR, by $C^{h,H}$ the coarsening operator from fine to coarse patches and by $I^{H,h}$ an interpolation operator of order q_I from coarse to fine subgrids. Finally, by $P^{h,H}$ we denote the restriction from a fine grid with mesh spacing h to a coarse grid with mesh spacing H and by Δ^h a discrete Laplacian, which in our case is L^h_{19} or L^h_{27}. After these preliminaries, the MLC method with two levels proceeds in four steps as follows:

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1. Downward pass

We loop over patches $\Omega^h_{R,\mathbf{i}}$ and using the local charges $f^{\mathbf{i},h}$ we compute the potentials $\phi^{\mathbf{i},h}$ in $\Omega^h_{R,\mathbf{i},\beta}$, that is

$$\phi^{\mathbf{i},h} = G^h * f^{\mathbf{i},h} \text{ in } \Omega^h_{R,\mathbf{i},\beta}. \tag{8}$$

For the computation of the local convolutions in (16) we are employing Hockney's domain doubling algorithm [7, 8]. For each $\Omega^h_{R,\mathbf{i}}$ we also compute the associated local charges

$$F^{\mathbf{i},H}[\boldsymbol{g}] = \begin{cases} \Delta^H P^{h,H} \phi^{\mathbf{i},h}[\boldsymbol{g}]\ ,\ \boldsymbol{g} \in C^{h,H}(\Omega^h_{R,\mathbf{i},\beta}) \\ \\ 0\ ,\ \boldsymbol{g} \notin C^{h,H}(\Omega^h_{R,\mathbf{i},\beta}) \end{cases} \tag{9}$$

2. Global Coarse Solve in Ω^H

Using the local charges $F^{\mathbf{i},H}$ we compute a global right hand side F^H

$$F^H = \sum_{\mathbf{i}} F^{\mathbf{i},H}$$

and compute the solution of the global problem $\Delta^H \phi^H = F^H$, in Ω^H:

$$\phi^H = G^H * F^H \text{ in } \Omega^H$$

For the global coarse solve we are also applying Hockney's algorithm.

3. Upward pass

At boundary points $\boldsymbol{g}$ of $\partial\Omega^h_{R,\mathbf{i}}$ we compute the following short-range local potentials as total contributions of potentials that have been computed on extended patches $\Omega^h_{R,\mathbf{i}',\beta}$ intersecting $\partial\Omega^h_{R,\mathbf{i}}$:

$$\phi^{loc,\boldsymbol{g}}[\boldsymbol{g'}] = \sum_{\mathbf{i}' : \boldsymbol{g} \in \Omega^h_{R,\mathbf{i}',\beta}} \phi^{\mathbf{i}',h}[\boldsymbol{g'}] \tag{10}$$

These local potentials are corrected by adding the contribution of far-field effects. The latter can be computed using the solution of the global coarse problem. Specifically, we have:

$$\phi^{B,\mathbf{i},h}[\boldsymbol{g}] = \phi^{loc,\boldsymbol{g}}[\boldsymbol{g}] + I^{H,h}\left(\phi^{H} - (\phi^{loc,\boldsymbol{g}})\right)(\boldsymbol{g}h),\ \boldsymbol{g} \in \partial\Omega^{h}_{R,\mathbf{i}}. \tag{11}$$

Details about the interpolation operator $I^{H,h}$ that we are employing can be found in [9]. Using the latter boundary conditions, we solve the following local Dirichlet problems on the $\Omega^{h}_{R,\mathbf{i}}$ patches using the fast Fourier transform [10]:

$$\begin{aligned} \Delta^{h}\tilde{\phi}^{MLC,\mathbf{i},h} &= f^{\mathbf{i},h} \text{ on } \Omega^{h}_{R,\mathbf{i}} - \partial\Omega^{h}_{R,\mathbf{i}} \\ \tilde{\phi}^{MLC,\mathbf{i},h} &= \phi^{B,\mathbf{i},h} \text{ on } \partial\Omega^{h}_{R,\mathbf{i}}. \end{aligned} \tag{12}$$

4. Post-processing step

Finally, we apply the fourth-order Mehrstellen correction to the solutions of the local Dirichlet problems and the values of $\phi^{MLC,h}$ in each patch $\Omega^{h}_{R,\mathbf{i}}$ are obtained:

$$\phi^{MLC,h}[\boldsymbol{g}] = \tilde{\phi}^{MLC,\mathbf{i},h}[\boldsymbol{g}] + \frac{1}{12}h^{2}f^{h}[\boldsymbol{g}]\ ,\ \boldsymbol{g} \in \Omega^{h}_{R,\mathbf{i}}. \tag{13}$$

It is noted that in the finest level of the original method, Hockney's algorithm is applied only to patches $\Omega^{h}_{R,\mathbf{i},\alpha}$ with $\alpha < \beta$. In the remaining annulus regions $\Omega^{h}_{R,\mathbf{i},\beta} \setminus \Omega^{h}_{R,\mathbf{i},\alpha}$ the local potentials are approximated with a discrete convolution of the Legendre projection of the local charge. Employing Legendre projections of order $P-1$ the *max-norm* error $e^{MLC,h}$ of the method is given by the estimate

$$e^{MLC,h} = O(h^{4}) + O\left(\frac{1}{\beta^{q}}h^{P}\right) + O\Big(h^{2}||f||_{\infty}\frac{1}{\beta^{q_{I}-2}}\Big) + O\Big(||f||_{\infty}\frac{1}{\beta^{q}}\Big). \tag{14}$$

As is evident, by appropriately choosing the order of the Legendre expansions and the parameter β the method is fourth order accurate in h and reaches a barrier error given by the last term in (14). The barrier error can be further reduced by applying larger values of β. Hence the error of the method is controlled not only by the mesh size but also from the degree of overlap among neighbor extended patches (parameter β) which can be thought of as an additional discretization parameter. To obtain sixth order accuracy and converge more rapidly to the barrier error we apply the sixth order scheme for the L^{h}_{27} Laplacian with the Mehrstellen correction of *type C*. The steps of the modified MLC method are as follows:

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0. Pre-processing step

At first we define the modified local charges:

$$f^{*,\mathbf{i},h} = f^{\mathbf{i},h} - \frac{h^{2}}{30}L^{h}_{7}f^{\mathbf{i},h} \tag{15}$$

1. Downward pass

Using Hockney's algorithm [7, 8] we solve for the local potentials $\phi^{*,\mathbf{i},h}$ in $\Omega^h_{R,\mathbf{i},\beta}$:

$$\phi^{*,\mathbf{i},h} = G^h_{27} * f^{*,\mathbf{i},h} \text{ in } \Omega^h_{R,\mathbf{i},\beta}. \tag{16}$$

The local charges in $\Omega^h_{R,\mathbf{i}}$ are computed by:

$$F^{*,\mathbf{i},H}[\boldsymbol{g}] = \begin{cases} L^H_{27} P^{h,H} \phi^{*,\mathbf{i},h}[\boldsymbol{g}] \text{ , } \boldsymbol{g} \in C^{h,H}(\Omega^h_{R,\mathbf{i},\beta}) \\ \\ 0 \text{ , } \boldsymbol{g} \notin C^{h,H}(\Omega^h_{R,\mathbf{i},\beta}) \end{cases} \tag{17}$$

2. Global Coarse Solve in Ω^H

A global right hand side $F^{*,H}$ is computed as follows:

$$F^{*,H} = \sum_{\mathbf{i}} F^{*,\mathbf{i},H}$$

The following global coarse problem $L^H_{27}\phi^{*,H} = F^{*,H}$, in Ω^H is solved by employing Hockney's algorithm:

$$\phi^{*,H} = G^H_{27} * F^{*,H} \text{ on } \Omega^H$$

3. Upward pass

Short-range local potentials are computed at boundary points $\boldsymbol{g}$ of $\partial\Omega^h_{R,\mathbf{i}}$:

$$\phi^{*,loc,\boldsymbol{g}}[\boldsymbol{g'}] = \sum_{\mathbf{i'}:\boldsymbol{g}\in\Omega^h_{R,\mathbf{i'},\beta}} \phi^{*,\mathbf{i'},h}[\boldsymbol{g'}] \tag{18}$$

and are modified to include the effect of long-range potentials as follows:

$$\phi^{*,B,\mathbf{i},h}[\boldsymbol{g}] = \phi^{*,loc,\boldsymbol{g}}[\boldsymbol{g}] + I^{H,h}\left(\phi^{*,H} - (\phi^{*,loc,\boldsymbol{g}})\right)(\boldsymbol{g}h), \; \boldsymbol{g} \in \partial\Omega^h_{R,\mathbf{i}}. \tag{19}$$

The local MLC potentials in $\Omega^h_{R,\mathbf{i}}$ are computed as solutions to the following Dirichlet problems via the fast Fourier transform [10]:

$$\begin{aligned} L^h_{27}\tilde{\phi}^{*,MLC,\mathbf{i},h} =& f^{*,\mathbf{i},h} \text{ on } \Omega^h_{R,\mathbf{i}} - \partial\Omega^h_{R,\mathbf{i}} \\ \tilde{\phi}^{*,MLC,\mathbf{i},h} =& \phi^{*,B,\mathbf{i},h} \text{ on } \partial\Omega^h_{R,\mathbf{i}}. \end{aligned} \tag{20}$$

4. Post-processing step

Further, the MLC local potentials are updated by means of the *type C* sixth-order Mehrstellen correction:

$$\phi^{MLC,h}[\boldsymbol{g}] = \tilde{\phi}^{*,MLC,\mathbf{i},h}[\boldsymbol{g}] + \frac{7h^2}{60} f^h[\boldsymbol{g}] - \frac{h^4}{240} L_7^h f^h[\boldsymbol{g}] \,,\, \boldsymbol{g} \in \Omega_{R,\mathbf{i}}^h. \tag{21}$$

With this modification of the MLC method it can be shown using the error analysis in [6] that the error satisfies the following estimate:

$$e^{MLC,h} = O(h^6) + O\left(\frac{1}{\alpha^q} h^P\right) + O\Big(h^2 ||f||_\infty \frac{1}{\beta^{q_I - 2}}\Big) + O\Big(||f||_\infty \frac{1}{\beta^q}\Big). \tag{22}$$

and the method converges to the barrier error at order six with respect to mesh size. We emphasize once more that the barrier error can be further reduced by increasing the value of parameter β .

In the sequel, we are applying the modified MLC method to solve the Poisson boundary value problem with infinite domain boundary conditions:

$$\begin{aligned} &\Delta\phi = f, \text{ in } \mathbb{R}^3, \\ &\phi(\boldsymbol{x}) = -\frac{1}{4\pi\|\boldsymbol{x}\|} \int_{\mathbb{R}^3} f(\boldsymbol{y}) d\boldsymbol{y} + o\left(\frac{1}{\|\boldsymbol{x}\|}\right), \;\; \|\boldsymbol{x}\| \to \infty, \end{aligned} \tag{23}$$

The computational domain is the unit cube $\Omega = [0,1]^3$. The charge function f is the superposition of three local charges $f_{\mathbf{c}_i}$ whose supports are spheres of radius $R_o = \frac{5}{100}$ centered at points $\boldsymbol{c}_1 = \left(\frac{3}{16}, \frac{7}{16}, \frac{13}{16}\right)$, $\boldsymbol{c}_2 = \left(\frac{7}{16}, \frac{13}{16}, \frac{3}{16}\right)$ and $\boldsymbol{c}_3 = \left(\frac{13}{16}, \frac{3}{16}, \frac{7}{16}\right)$:

$$f_{\boldsymbol{c}_i}(\boldsymbol{x}) = \begin{cases} \frac{1}{R_o^3}(r - r^2)^2 \sin^2(\frac{\gamma}{2} r), r < 1 \\ \qquad\qquad\qquad\qquad 0, r \geq 1 \end{cases} , r = \frac{1}{R_o}\|\boldsymbol{x} - \boldsymbol{c}_i\|, \gamma = 4\mu\pi, \mu = 7 \tag{24}$$

The exact solution of (23) is the superposition of the following potentials:

$$\phi_{\boldsymbol{c}_i}(\boldsymbol{x}) = \frac{1}{R_o} \begin{cases} -\frac{1}{120} - \frac{6}{\gamma^4} & , \;\; r = 0 \\ \frac{r^6}{84} - \frac{r^5}{30} + \frac{r^4}{40} + \frac{60}{\gamma^6} - \frac{9}{\gamma^4} - \frac{1}{120} + \frac{120}{\gamma^6 r} & \\ + \left(-\frac{120}{\gamma^6 r} - \frac{9}{\gamma^4} + \frac{300}{\gamma^6} + \frac{36r}{\gamma^4} + \frac{r^2}{2\gamma^2} - \frac{30r^2}{\gamma^4} - \frac{r^3}{\gamma^2} + \frac{r^4}{2\gamma^2}\right) \cos(\gamma r) & \\ + \left(\frac{12}{\gamma^5 r} - \frac{360}{\gamma^7 r} - \frac{96}{\gamma^5} + \frac{120r}{\gamma^5} - \frac{3r}{\gamma^3} + \frac{8r^2}{\gamma^3} - \frac{5r^3}{\gamma^3}\right) sin(\gamma r) & , \;\; r < 1 \\ \left(-\frac{1}{210} - \frac{12}{\gamma^4} + \frac{360}{\gamma^6}\right) \frac{1}{r} & , \;\; r \geq 1 \end{cases}$$

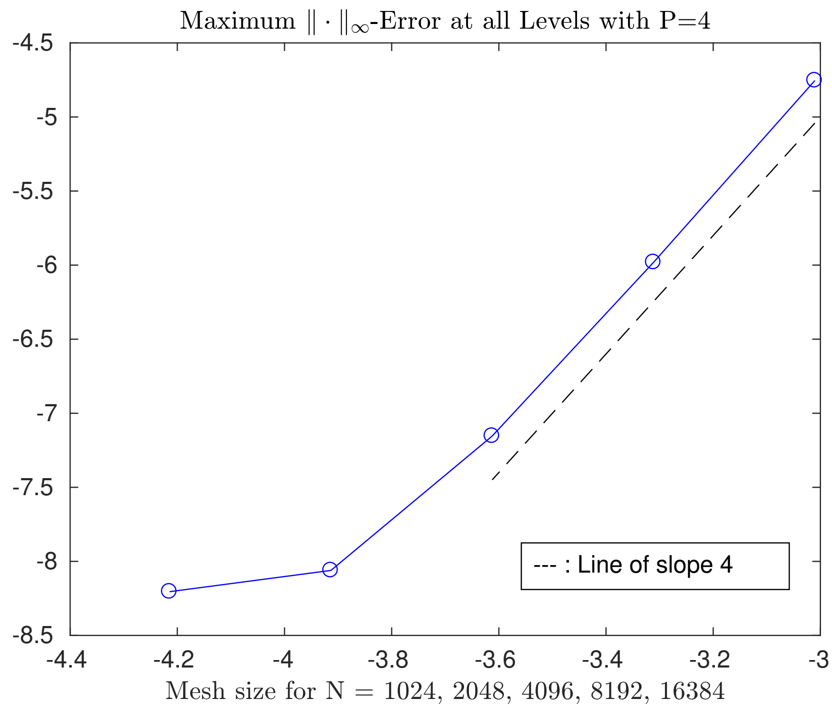

Figure 1: $Log_{10} - log_{10}$ plot of greatest max norm error at all levels against mesh size using the L^h_{27} Laplacian with fourth order Mehrstellen correction and 4 levels [6]. Here fourth order Legendre polynomials are employed at level 3.

The refinement ratio between successive coarse and fine levels is *4* in the test case we are presenting and each grid $\Omega^h_{R,\mathbf{i}}$ has N^3 cells where $N = 32$. The mesh sizes we are considering are $h = \frac{1}{1024}, \frac{1}{2048}, \frac{1}{4096}, \frac{1}{8192}, \frac{1}{16384}$. The value of parameter β is *3.25*. At the finest level (*level 3*) we are employing Legendre expansions of order *4* and set $\alpha = 2.25$. The adaptive grid for the $h = \frac{1}{16384}$ case comprises 7.5 billion elements. For the case of the fourth order Mehrstellen correction [6] we present in Figure 1 a *log-log* plot of the scaled *max-norm* error that is given by:

$$\frac{\|\phi^{MLC,h} - \phi\|_\infty}{\|\phi\|_\infty} \tag{25}$$

As is evident, the method is fourth order accurate and the barrier error is reached at finest mesh resolutions. The results obtained with the sixth order Mehrstellen correction are depicted in Figure 2. In this case the method is sixth order accurate and approaches faster the barrier error. Our implementation in this work is based on the Chombo scientific computing library [11]. For the computation of fast Fourier transforms we have employed the FFTW library [12].

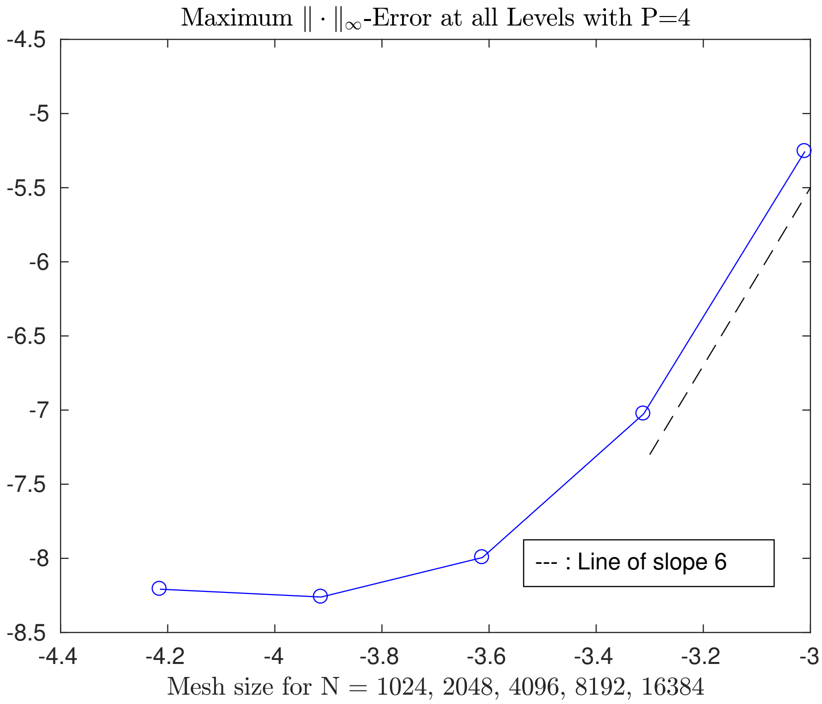

Figure 2: $Log_{10} - log_{10}$ plot of greatest max norm error at all levels against mesh size using the L^h_{27} Laplacian with sixth order Mehrstellen correction and 4 levels. Here fourth order Legendre polynomials are employed at level 3.

4 CONCLUSIONS

We have presented a sixth order compact finite difference scheme for Poisson's equation in three dimensions. The scheme has been applied to the MLC method [6] to further improve its convergence. Extension to the finite volume case has been worked out in [13] for the L^h_{19} operator. Future work will consider high order compact stencils for other fundamental differential operators such as the Helmholtz, biharmonic, Cauchy-Navier and Stokes operators and analogous extensions of the MLC method.

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