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TOPOLOGY OPTIMIZATION SUBJECT TO CONSTRAINTS ON THE INTERFACE STIFFNESS MATRIX

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Summary. Topology optimization generates optimal designs for specified loads. This includes designs with minimal mass subject to a constraint on a system response, such as deformation. Many loading scenarios, and, thus, many constraints on the system response can make topology optimization expensive. This is relevant, e.g., under uncertain loading conditions. Similarly, in optimization problems subject to constraints on the total eigenfrequency, many vibrational modes have to be considered.

For these problems, it is beneficial to formulate constraints on the stiffness of the structure rather than on its response to individual loads. In the approach introduced by Frank et al. [1] and Frank and Zimmermann [2], this is done using a so-called interface stiffness matrix (or similarly its inverse, an interface compliance matrix). More specifically, the lowest eigenvalue λ_{min} of the so-called stiffness excess matrix, i.e., the difference between the interface stiffness matrix $\mathbf{K}$ and a limit matrix $\mathbf{K}_c$, are subject to the constraint to be positive to ensure sufficient overall stiffness.

In this paper we present (1) a motivation of a meaningful limit stiffness matrix $\mathbf{K}_c$ for many loads, and (2) an algorithm to solve the topology optimization problem subject to the constraint $\lambda_{min}(\mathbf{K} - \mathbf{K}_c) \geq 0$ to ensure sufficient stiffness.

1 INTRODUCTION

Structural optimization plays a crucial role in achieving lightweight design goals by optimizing the distribution of materials in order to maximize structural efficiency, typically reflected in improved stiffness-to-weight ratios. System requirements typically include displacement, stress, or compliance constraints for specific load cases. Optimization is computationally expensive when multiple loading scenarios or uncertain loading conditions are considered. In contrast, the stiffness matrix, which contains the structure's geometric and material characteristics, is independent of the loading conditions. By formulating constraints directly on the stiffness matrix by setting a limit stiffness matrix, the optimization can be effectively decoupled from the loading scenarios. This approach reduces the computational cost by eliminating the need to address each load case individually and enhances the flexibility of the design process. Krischer and Zimmermann [3] successfully applied this principle by employing so-called informed decomposition to break down complex optimization problems into manageable components. They used spe-

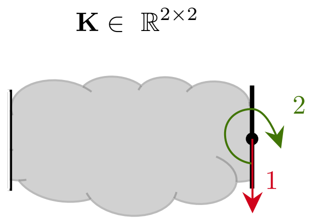


Figure 1: Design domain with a rigid interface with two degrees of freedom

cific entries of the interface stiffness matrix and topology optimization to develop meta-models that predict mass and feasibility, which facilitated the assignment of target stiffness values to individual components.

This work addresses both establishing a limit on the interface stiffness matrix and the mass minimization based on that limit stiffness matrix.

The advantage of using interface stiffness matrices as optimization constraints lies in its load independence during the topology optimization. This makes it particularly suitable for scenarios with uncertain loading conditions, which can be addressed using sampling strategies (e.g., [4, 5, 6]). Additionally, it is effective for optimization involving a wide range of loads e.g. from dynamic cases [7] or other uncertain loading scenarios [8, 9]. Requirement formulations on stiffness entries can also be used for optimization of larger systems where system requirements are to be broken down on component level as demonstrated by Krischer [3, 10].

The constraints are set on the eigenvalues of the stiffness excess matrix, i.e., the difference between the interface stiffness matrix $\mathbf{K}$ and the limit matrix $\mathbf{K}_c$. In literature, eigenvalues are often involved in structural optimization, particularly when manipulating the natural frequencies of a system to avoid resonance or prevent buckling [11, 12]. These applications highlight the importance of eigenvalues in determining structural behavior under various operational conditions. In the present work, eigenvalues are not used to address dynamic or failure problems, but to formulate requirements on the stiffness of components.

This work explores the implementation for the minimum eigenvalue constraint with respect to the interface stiffness matrix based on [13]. The method directly imposes constraints on the eigenvalues of the interface excess matrix, while the eigenvalues and eigenvectors are tracked, meaning that the order of eigenvalues and eigenvectors does not change. This ensure differentiability for repeated eigenvalues. Topology optimization will generate structures that meet stiffness requirements efficiently.

2 METHODS AND PROBLEM STATEMENTS

2.1 Setup

To demonstrate the concept, a simple rectangular design domain Ω is used. The structure is fixed on the left side and features a rigid interface on the right, as shown in Figure 1. The load is introduced at the interface node as $\mathbf{f} = [F, M]^T$, which possesses one translational and one rotational degree of freedom. These are represented in the interface displacement vector $\mathbf{q} = [u, \varphi]^T$. Using linear elastic material behavior, the response of the structure can be modelled by

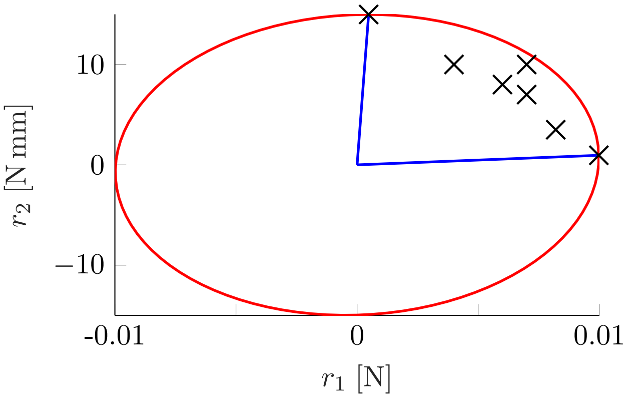


Figure 2: Visualization of $\tilde{\mathbf{C}}_c$ enclosing all directional stiffness vectors $\mathbf{r}_j$.

$$\mathbf{Kq} = \mathbf{f}. \tag{1}$$

2.2 Problem statement

Starting point is a structural design optimization problem for design variables $\mathbf{x}$, which may be nodal densities $\boldsymbol{\rho}$ for topology optimization, as presented in Equation 2, where a constraint l_{jc} is imposed on the compliance l_j of the interface,

$$\begin{aligned} \min_{\mathbf{x}} \quad & m(\mathbf{x}) \\ \text{s.t.} \quad & l_j \leq l_{jc} \quad \text{for} \quad j = 1, \ldots, n_{lc}. \end{aligned} \tag{2}$$

The problem can be split into two parts: The first part identifies a minimum limit stiffness matrix, ensuring that the displacement constraints are fulfilled for all load cases j

$$\begin{aligned} \min_{\mathbf{K}_c} \quad & ||\mathbf{K}_c|| \\ \text{s.t.} \quad & l_j \leq l_{jc} \quad \text{for} \quad \mathbf{K}_c\mathbf{q}_j = \mathbf{f}_j. \end{aligned} \tag{3}$$

The second part is then given by

$$\begin{aligned} \min_{\mathbf{x}} \quad & m(\mathbf{x}) \\ \text{s.t.} \quad & \lambda_i(\mathbf{K}(\mathbf{x}) - \mathbf{K}_c) \geq 0, \quad i = 1, \ldots, n_{dim}, \end{aligned} \tag{4}$$

where λ_i denote the eigenvalues of the matrix argument. This works because the inequality constraint implies that for any arbitrary vector $\mathbf{q}$, the inequality $\mathbf{q}^T\mathbf{Kq} \geq \mathbf{q}^T\mathbf{K}_c\mathbf{q}$ must hold. Now, if we choose $\mathbf{q} = \mathbf{q}_j$, this leads to $\mathbf{f}_j^T\mathbf{q}_j = \mathbf{q}_j^T\mathbf{K}_c\mathbf{q}_j = l_j$, where $l_j \leq l_{jc}$. This means that the constraint of the initial problem is satisfied.

The construction of $\mathbf{K}_c$ is described in the following section.

2.3 Constructing $\mathbf{K}_c$

We seek the smallest $\mathbf{K}_c$, or equivalently, the largest limit compliance matrix $\mathbf{C}_c$ that still satisfies the condition

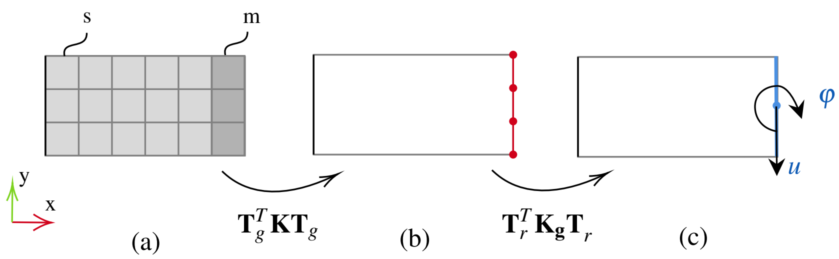


Figure 3: Design domain is discretized by finite elements, (b) slave nodes are removed with Guyan reduction, and (c) remaining nodes are connected rigidly to the interface node with an RBE2 element.

$$\mathbf{f}_j^T \mathbf{C}_c \mathbf{f}_j \leq l_{jc} \quad \forall j, \quad \text{where} \;\; \mathbf{C}_c = \mathbf{K}_c^{-1}. \tag{5}$$

First, force vectors are normalized by the compliance limits, yielding

$$\mathbf{r}_j = \frac{\mathbf{f}_j}{\sqrt{\frac{l_{jc}}{El^3}}}. \tag{6}$$

This way, Equation 5 can be replaced by

$$\mathbf{r}_j^T \tilde{\mathbf{C}}_c \mathbf{r}_j \leq 1, \quad \text{with} \;\; \tilde{\mathbf{C}}_c = \frac{1}{El^3}\mathbf{C}_c. \tag{7}$$

The solution of Equation 7 satisfies the initial constraint of the optimization problem as long as the constraint in expression 4 is satisfied. In two dimensions, the equation can be represented geometrically as an ellipse. In Figure 2 the red ellipse shows all $\mathbf{r}_j$ for which $\mathbf{r}_j^T \tilde{\mathbf{C}}_c \mathbf{r}_j = 1$ and is, thus, a visualization of the magnitude of $\tilde{\mathbf{C}}_c$. The larger $\tilde{\mathbf{C}}_c$, the smaller is the ellipse. Once, $\tilde{\mathbf{C}}_c$ encloses all load vector $\mathbf{r}_j$, Equation 5 is satisfied. This problem is treated, e.g., in [15]. Imposing the constraint $\mathbf{f}_j^T \mathbf{C} \mathbf{f}_j \leq \mathbf{f}_j^T \mathbf{C}_c \mathbf{f}_j \quad \forall \mathbf{f}_j$ will ensure that the constraint in Equation 5 is satisfied. For invertible $\mathbf{C}_c$ using $\mathbf{C} \leq \mathbf{C}_c \Leftrightarrow \mathbf{K} \geq \mathbf{K}_c$, we finally obtain the limit stiffness matrix $\mathbf{K}_c$.

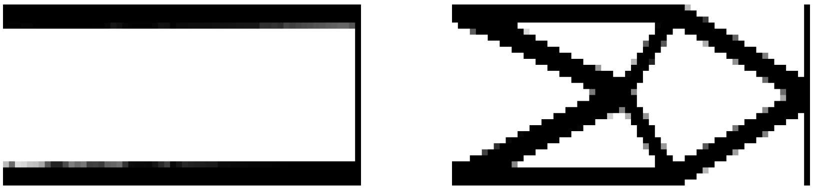
The advantage of using the limit stiffness matrix over the limit compliance matrix lies in its simplicity. Handling zero stiffness values is more stable compared to managing infinite entries in the compliance matrix. Furthermore, the limit stiffness matrix enables zero entries in directions where stiffness is not required, making it more versatile for application.

2.4 Topology optimization

For topology optimization with SIMP, a finite element model of the system is required. Therefore, the domain is discretized using 60 × 30 quadrilateral plane stress elements. The finite element model includes a large nodal stiffness matrix, which requires a two-step reduction process for condensation on the interface node. First, the right-most nodes of the structure are treated as slave degrees of freedom using Guyan reduction. Then, these nodes are connected to a rigid interface, as shown in Figure 3. The interface stiffness matrix $\mathbf{K}$ is calculated as follows:

$$\mathbf{K} = \mathbf{T}_r^T \mathbf{T}_g^{\mathrm{T}} \mathbf{K}_{FE} \mathbf{T}_g \mathbf{T}_r, \tag{8}$$

where $\mathbf{K}_{FE}$ represents the nodal stiffness matrix and $\mathbf{T}_r$ and $\mathbf{T}_g$ including the static condensation for the rigid interface and onto the interface node, respectively.



(B1) (B2)

Figure 4: Density distribution of the optimized structure for benchmark cases. B1 corresponds to $\mathbf{K}_c = \begin{pmatrix} 0 & 0 \\ 0 & 20\,\mathrm{N\,mm} \end{pmatrix}$ and B2 to $\mathbf{K}_c = \begin{pmatrix} 0.01\,\mathrm{N\,mm}^{-1} & 0 \\ 0 & 0 \end{pmatrix}$

The structural behavior can be fully described by the derived 2×2 stiffness matrix. This allows the prescription of the interface stiffness properties by a limit matrix used for a constraint.

Topology optimization is carried out using the Solid Isotropic Material with Penalization (SIMP) method, applying a penalty factor of 3. A density filter with a minimum radius $r_{min} = \sqrt{2}\,\mathrm{mm}$ is employed to prevent checkerboard patterns. All elements are initialized with a uniform density distribution at a 50% volume fraction. The Method of Moving Asymptotes (MMA) is used to solve the optimization problems, with algorithmic hyperparameters set according to the guidelines provided by Svanberg in [14].

3 TOPOLOGY OPTIMIZATION

The gradient of the constraint function is calculated based on the free vibration example of Bendsøe and Sigmund [16]. The eigenvalue problem for this scenario is defined as

$$(\mathbf{D} - \lambda_i \mathbf{I})\mathbf{v}_i = 0, \tag{9}$$

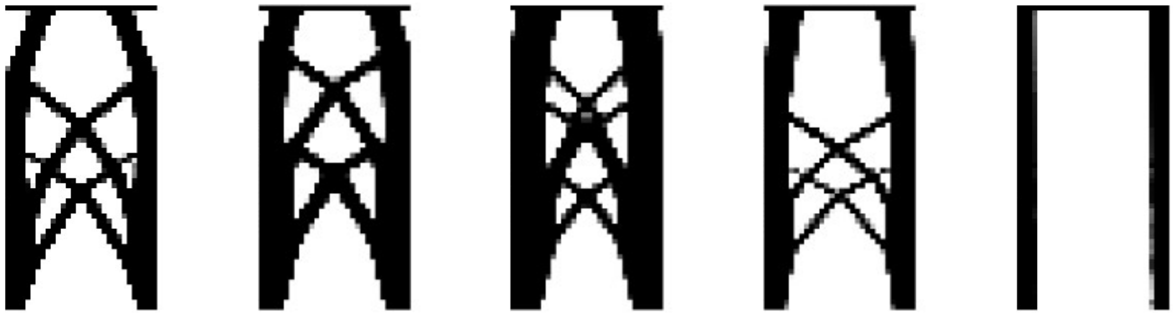
where $\mathbf{D} = \mathbf{K} - \mathbf{K}_c$ is the so-called stiffness excess matrix and $\mathbf{v}_i$ is the eigenvector corresponding to the eigenvalue λ_i. The gradient of λ_i with respect to the design variables ρ_e is derived to

$$\frac{\partial \lambda_i}{\partial \rho_e} = \mathbf{v}_i^{\mathrm{T}} \frac{\partial \mathbf{K}(\rho_e)}{\partial \rho_e} \mathbf{v}_i = \mathbf{v}_i^T \mathbf{T}_g^T \frac{\partial \mathbf{K}(\rho_e)}{\partial \rho_e} \mathbf{T}_g \mathbf{v}_i = p \rho_e^{p-1} \mathbf{v}^T \mathbf{T}_g^T \mathbf{K}_e \mathbf{T}_g \mathbf{v}. \tag{10}$$

For topology optimization with finite elements, the problem reads now

$$\begin{aligned} \min_{\rho_e} \quad & m(\rho_e) \\ \text{s.t.} \quad & \lambda_i(\mathbf{D}(\rho_e)) \geq 0, \quad i = 1, \ldots, n_{dim} \\ & 0 \leq \rho_e \leq 1, \qquad\quad e = 1, \ldots, n. \end{aligned} \tag{11}$$

To verify the correctness of the implementation, two benchmark cases were employed: (B1) for pure bending moment and (B2) for pure shear force. These cases are chosen due to their simplicity and the predictable outcomes derived from standard sample problems in topology optimization. The resulting density distributions for the minimum eigenvalue constraint are shown in Figure 4. For case (B1), the theory predicts maximum normal stress at the upper and lower boundaries of the domain. Correspondingly, the density distribution reveals beam-like structures forming at these boundaries, aligning with theoretical expectations. For case (B2),



$\mathbf{K}_c = \begin{bmatrix} 0.01 & 0 \\ 0 & 10 \end{bmatrix}$ $\mathbf{K}_c = \begin{bmatrix} 0.01 & 0 \\ 0 & 15 \end{bmatrix}$ $\mathbf{K}_c = \begin{bmatrix} 0.01 & 0 \\ 0 & 20 \end{bmatrix}$ $\mathbf{K}_c = \begin{bmatrix} 0.005 & 0 \\ 0 & 20 \end{bmatrix}$ $\mathbf{K}_c = \begin{bmatrix} 0 & 0 \\ 0 & 20 \end{bmatrix}$

$\mathbf{K}: \begin{bmatrix} 0.011 & 0.027 \\ 0.027 & 11.634 \end{bmatrix}$ $\begin{bmatrix} 0.015 & 0.224 \\ 0.224 & 25.392 \end{bmatrix}$ $\begin{bmatrix} 0.022 & 0.546 \\ 0.546 & 44.615 \end{bmatrix}$ $\begin{bmatrix} 0.008 & 0.121 \\ 0.121 & 25.651 \end{bmatrix}$ $\begin{bmatrix} 0.001 & 0.010 \\ 0.010 & 20.225 \end{bmatrix}$

Figure 5: Topology optimization results for different limit stiffness matrices $\mathbf{K}_c$ and the realized stiffness matrices $\mathbf{K}$. Units are $[\mathbf{K}] = \begin{pmatrix} \mathrm{N\,mm^{-1}} & \mathrm{N} \\ \mathrm{N} & \mathrm{N\,mm} \end{pmatrix}$.

the stress distribution increases towards the support, manifesting a typical cantilever design, which also accords with the expected outcome.

Figure 5 displays a selection of topology optimization results for various limit stiffness matrices. As observed in the benchmark cases, an increase in the k_{22} entry results in thicker structures at the boundary of the domain, whereas a decrease in the k_{11} entry gradually removes the structure in the middle of the domain. For the examples presented, the eigenvalues of the stiffness excess matrix exhibit similar behavior. Specifically, the smaller eigenvalue λ_1 approaches zero, while the larger eigenvalue λ_2 remains greater than zero. In scenarios where the two eigenvalues do not both approach zero, the optimization process tends to converge effectively, producing stable and feasible designs.

However, when off-diagonal terms are included in the limit stiffness matrix, the optimization landscape changes significantly. In such cases, the limit stiffness matrix can be exactly matched by a structure, leading to a stiffness excess matrix with all zero entries. This situation causes both eigenvalues to become zero, which creates convergence problems for the optimization algorithm. When both eigenvalues approach zero, one of the constraints is violated by MMA, resulting in difficulties for the algorithm to find a feasible solution. Therefore, the current approach is primarily useful in scenarios where this specific issue of both eigenvalues becoming zero cannot occur.

The implementation of the eigenvalue constraints requires careful handling to avoid problems associated with eigenvalue ordering. In the constraint functions, it is not sufficient to only use the smallest eigenvalue as described in the initial problem formulation. This approach can lead to complications when the order of the eigenvalues switches during the optimization process. To address this, constraints are imposed on both eigenvalues, ensuring that they are each larger than or equal to zero. This results in two separate constraint functions being incorporated into the optimization problem. The eigenvalues and their corresponding eigenvectors are tracked throughout the optimization process. Tracking helps to prevent the switching of eigenvalue order, which can disrupt the convergence of the algorithm. While this approach makes the convergence process smoother, there are still instances where a constraint becomes temporarily violated, leading to oscillations. These oscillations can cause the algorithm to struggle to maintain feasibility, resulting in a less stable convergence process.

4 CONCLUSION

This study delves into topology optimization subject to constraints on the interface stiffness matrix. A method was proposed that decouples the optimization process from specific loading scenarios. The proposed approach ensures sufficient stiffness for multiple load cases by specifying a limit stiffness matrix, which is used for constraint formulation in the subsequent mass minimization process.

Formulating constraints on the stiffness matrix, rather than on individual load responses, offers significant advantages, especially in handling uncertain or numerous loading conditions. This method was demonstrated on benchmark cases, aligning with theoretical expectations. However, the study also highlights persistent challenges such as repeated eigenvalues and eigenvalue switching, which can hinder the convergence of gradient-based optimization algorithms, confirming issues identified in prior research in [17, 18, 19].

Overall, the proposed methodology provides a framework for optimizing lightweight structures. Open questions for further research include managing eigenvalue constraints to prevent convergence issues with specific limit stiffness matrices.

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