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COLLOCATION AND GALERKIN ISOGEOMETRIC APPROXIMATIONS OF ACOUSTIC WAVES: A NUMERICAL COMPARISON OF SPECTRAL PROPERTIES

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Summary. The acoustic wave equation with absorbing boundary conditions is approximated using collocation and Galerkin Isogeometric analysis (IGA) in space and implicit second-order Newmark schemes in time. We report numerical results for the condition number of the mass and iteration IGA matrices, in particular we study their dependence on the polynomial degree p, mesh size h and regularity k. The results show that the spectral properties of the IGA collocation matrices are analogous or in some cases better than the corresponding IGA Galerkin that are known for the Poisson problem with Dirichlet boundary conditions and are also studied experimentally here.

1 INTRODUCTION

Isogeometric analysis (IGA) has been successfully used on a large variety of areas since its introduction in [13], with important results in several practical problems involving the numerical solution of partial differential equations, see, e.g., [1, 3, 4, 5], and the references therein. In the framework of IGA approximation, several works have recently focused on explicit acoustic and elastic waves, using both Galerkin, Discontinuous Galerkin and collocation methods [2, 6, 9, 14, 16, 27, 28, 29]. IGA methods are based on the choice of the same functions to construct the CAD geometry and to represent the approximated solution of the PDE. In this way, IGA yields an exact representation of the computational domain and at the same time a higher order approximation error with respect to standard p- and hp- refinements, where p is the polynomial degree of the IGA basis functions and h is the mesh size of the elements. IGA also enables an additional k-refinement, where $k \leq p-1$ is the global regularity of the IGA basis functions. While initially IGA works have been carried out using Galerkin approaches, more recently IGA collocation methods have been largely investigated, with the aim of dealing with sparser mass and stiffness matrices than those arising from IGA Galerkin variants. IGA collocation has also the additional advantage of reducing the global computational cost, since collocation matrices require only one function evaluation per collocation point, independently of p; see [2, 9, 16].
In our previous works [27]-[29] we have studied and compared the stability and convergence properties of IGA approximations of the acoustic wave equation with first order absorbing boundary conditions, using Newmark's schemes in time. Since both the IGA collocation and Galerkin mass matrices are not diagonal, the solution of the linear systems at each time step is a crucial point for both explicit and implicit Newmark schemes. Unfortunately, there is still a lack of theoretical results for IGA matrices' properties in the literature, particularly in the case of IGA collocation,

and most of the known estimates on conditioning of the IGA mass and stiffness matrices are actually conjectures. See, e.g. [10, 11, 12] for some estimates and numerical results regarding IGA Galerkin matrices associated to the Poisson problem with Dirichlet boundary conditions. In absence of theoretical bounds for condition numbers of the IGA mass and stiffness matrices particularly for wave problems, in [30] we have carried out an extensive numerical investigation of the spectral properties of the IGA matrices in the framework of collocation.
The main novelty of this paper is to consider also the spectral properties of IGA Galerkin matrices for the acoustic wave equation with first order absorbing boundary conditions, and to present a direct numerical comparison of collocation and Galerkin IGA methods with respect to the condition numbers of their mass and iteration matrices for acoustic wave problems with absorbing boundary conditions, discretized with Newmark methods in time, varying the polynomial degree p, mesh size h, regularity k. Our numerical results show that the same trends hold for the condition numbers of the IGA collocation and Galerkin mass and stiffness matrices with respect to the mesh size h, and these results are comparable to those available in [10, 12] for the Poisson problem. If we consider instead the behaviour of the condition numbers for increasing values of the degree p, we observe that they are better for the IGA collocation case than for IGA Galerkin, both for minimal and maximal regularity k.
The rest of the paper is organized as follows. The acoustic wave model problem is introduced in Section 2. In Section 3 we describe its approximation by IGA collocation and Galerkin in space and by Newmark scheme in time. Finally, in Section 4, after a brief overview of the existing literature for eigenvalue and condition number estimates of IGA Galerkin approximation of the Poisson problem, we present several numerical tests on the comparison of condition numbers of IGA collocation and Galerkin mass and iteration matrices, varying all the discretization parameters.

2 The acoustic wave model problem

Let Ω be a finite region in the plane, $\partial\Omega$ is its boundary and $\mathbf{n}$ is the outward normal unit vector. We consider the acoustic wave problem (see e.g., [15]):

$$\frac{\partial^2 u}{\partial t^2}(\mathbf{x}, t) - c_0 \Delta u = f(\mathbf{x}, t) \quad \text{in } \Omega \times (0, T), \tag{1}$$

with initial conditions

$$u(\mathbf{x}, 0) = \mathcal{U}_0(\mathbf{x}), \qquad \frac{\partial u}{\partial t}(\mathbf{x}, 0) = \mathcal{W}_0(\mathbf{x}) \qquad \text{in } \Omega. \tag{2}$$

In the above equations, c_0 is the acoustic wave propagation velocity, f is the source term, $\mathcal{U}_0$ and $\mathcal{W}_0$ are the initial pressure and velocity, respectively, u is the unknown pressure, $\mathbf{x}$ is any point of Ω, t is the time variable, $(0, T)$ is the temporal interval, with $T \in \mathbb{R}^+$. We make the assumption that, for each $t \in (0, T)$, $f \in L^2(\Omega \times (0, T))$, $\mathcal{U}_0 \in H^1(\Omega)$ and $\mathcal{W}_0 \in L^2(\Omega)$. For the definition of the above boundary spaces we refer, e.g., to [17], vol. I. Since wave propagation problems are usually set in unbounded domains, one of the most common strategy in order to refer to a finite domain Ω in (1) is to enforce absorbing boundary conditions (ABCs for brevity) rather than standard Dirichlet or Neumann ones. In this regard, we consider here natural first-order ABCs involving first spatial and temporal partial derivatives only (see, e.g. [19, 21]):

$$\frac{1}{\sqrt{c_0}} \frac{\partial u}{\partial t}(\mathbf{x}, t) + \frac{\partial u}{\partial \mathbf{n}}(\mathbf{x}, t) = 0 \qquad \text{on } \partial\Omega \times (0, T), \tag{3}$$

that enable to truncate the original unbounded domain into a finite one, thus reducing spurious wave reflections as possible. The weak form of (1)-(3) reads: find $u : (0,T) \to V \equiv H^1(\Omega)$, such that for *a.e.* $t \in (0,T)$:

$$\left(\frac{\partial^2 u}{\partial t^2}, v\right) + a(u,v) + \sqrt{c_0} < \frac{\partial u}{\partial t}, v >_{\partial\Omega} = (f,v) \quad \forall v \in V, \tag{4}$$

$$a(u,v) = c_0 \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}\mathbf{x}, \quad (f,v) = \int_\Omega f v \, \mathrm{d}\mathbf{x}, \ < \zeta, v >_{\partial\Omega} = \int_{\partial\Omega} \zeta v \, \mathrm{d}s, \tag{5}$$

where the bilinear form $a(\cdot,\cdot)$ is symmetric, V-elliptic and continuous. For the proof of stability and uniqueness of the continuous acoustic problem in the case of ABCs, we refer to [21] where the similar case of elastodynamics linear problems is studied.

3 B-splines and NURBS-based Isogeometric Analysis

We briefly recall the semidiscrete continuos-in-time numerical approximation of the acoustic wave problem in the strong (1-3) and weak (4) forms, that are based on IGA collocation and Galerkin, respectively. Given an open *knot vector*

$$\{\xi_1 = 0, ..., \xi_{\nu+p+1} = 1\}, \tag{6}$$

of non-decreasing real numbers in the reference interval $[0,1]$, we consider univariate B-spline basis functions N_i^p having support (ξ_i, ξ_{i+p+1}), $i = 1, 2, ..., \nu$, where p is the polynomial degree and ν is the number of basis functions and control points. Starting from piecewise constant functions corresponding to degree $p = 0$, then B-splines are built recursively, see, e.g. [25]. If internal nodes are not repeated, B-spline basis functions are C^{p-1}-continuous, whereas if the associated knot has multiplicity equal to α, the basis is C^k-continuous, with $k = p - \alpha$. In particular, where a knot has multiplicity $\alpha = p$, the basis is C^0-continuous. For all considered functions to be at least globally continuous, we assume that the maximum knot multiplicity is p. For the sake of simplicity, we take the case of the same degree p in each direction. The case of different degrees can be defined analogously. We denote by $\widehat{\Omega} := (0,1) \times (0,1)$ the two-dimensional parametric space built from a knot vector (6) in each direction, and $\mathbf{C}_{i,j}$ is the net of ν^2 control points, $i, j = 1, ..., \nu$. Similarly, the multi-dimensional B-spline basis functions on $\widehat{\Omega}$ are obtained by tensor product as $B_{i,j}^p(\xi,\eta) = N_i^p(\xi) N_j^p(\eta), \ i,j = 1, \ldots, \nu$. Finally, given the one-dimensional spline space $\mathrm{span}\{N_i^p(\xi), i = 1, \ldots, \nu\}$, we obtain the bi-variate spline space:

$$\widehat{\mathcal{S}}_h = \mathrm{span}\{B_{i,j}^p(\xi,\eta), \ i,j = 1, \ldots, \nu\}. \tag{7}$$

See [23] and references therein for details. We introduce a NURBS basis function of degree p as

$$R_i^p(\xi) = \frac{N_i^p(\xi)\omega_i}{w(\xi)}, \tag{8}$$

where $w(\xi) = \sum_{\hat{i}=1}^{\nu} N_{\hat{i}}^p(\xi)\omega_{\hat{i}} \in \widehat{\mathcal{S}}_h$ is a weight function, and we build NURBS basis functions on the two-dimensional parametric space $\widehat{\Omega}$ as

$$R_{i,j}^p(\xi,\eta) = \frac{B_{i,j}^p(\xi,\eta)\omega_{i,j}}{w(\xi,\eta)}, \tag{9}$$

where $w(\xi,\eta) = \sum_{\hat{i},\hat{j}=1}^{\nu} B_{\hat{i},\hat{j}}^p(\xi,\eta)\omega_{\hat{i},\hat{j}}$ and $\omega_{i,j} \in \mathbb{R}$ is the two-dimensional weight. We recall that NURBS basis functions obtained by the span of the basis functions (9) have the same continuity and support of B-splines. If we confine ourselves to the case of a *single-patch* domain Ω as a NURBS region associated with the net $\mathbf{C}_{i,j}$, we can define the geometrical map $\mathbf{F}:\widehat{\Omega}\to\Omega$:

$$\mathbf{F}(\xi,\eta)=\sum_{i,j=1}^{\nu} R^p_{i,j}(\xi,\eta)\mathbf{C}_{i,j}. \tag{10}$$

According to the IGA paradigm, the span of the *push-forward* of the basis functions (9) provides the space of NURBS scalar fields on the domain Ω:

$$\mathcal{N}_h := \mathrm{span}\{R^p_{i,j}\circ\mathbf{F}^{-1},\ \text{with } i,j=1,\dots,\nu\}. \tag{11}$$

3.1 IGA collocation discretization of the acoustic problem

In this section we briefly review the IGA collocation method, see [1, 2, 24], and apply it to our acoustic wave problem with ABCs. We choose as collocation points the classical Greville abscissae $\overline{\xi}_i \doteq (\xi_{i+1}+\xi_{i+2}+...+\xi_{i+p})/p$, $i=1,...,\nu$, associated with the given knot vector (6), where $\overline{\xi}_1=0$, $\overline{\xi}_\nu=1$, with the remaining points in $(0,1)$. See [8]. Other choices of isogeometric collocation abscissae have been proposed, see e.g. [22]. Then we define the grid of collocation points $\tau_{ij}\in\Omega$ by the tensor product $\tau_{ij}=\mathbf{F}(\widehat{\tau}_{ij})$, $\widehat{\tau}_{ij}=(\overline{\xi}_i,\overline{\xi}_j)\in\overline{(\widehat{\Omega})}$, $i,j=1,...,\nu$. For a more straightforward description of the collocation method, it is convenient to enumerate the grid points $\{\tau_{ij}\}$ using one single index. Thus each collocation point $\tau_{ij}\in\Omega, i,j=1,...,\nu$, corresponds to the point P_ℓ of the tensor product grid, with $\ell=1,...,\nu^2$. We also introduce two disjoint sets of indexes $\mathcal{I}_\Omega := \{\ell|P_\ell\in\Omega\}$ and $\mathcal{I}_{\partial\Omega} := \{\ell|P_\ell\in\partial\Omega\}$, associated to internal and boundary points, respectively. We denote by $\mathcal{I}:=\mathcal{I}_\Omega\cup\mathcal{I}_{\partial\Omega}$ the set of ν^2 indexes of the whole grid of mesh points, and we obtain the IGA collocation, semi-discrete continuous-in-time approximation of the acoustic problem (1)-(3) by imposing the continuous problem (1) at the Greville collocation points:

$$\frac{\partial^2 u}{\partial t^2}(P_\ell,t)-c_0\Delta u(P_\ell,t)=f(P_\ell,t),\quad \ell\in\mathcal{I}_\Omega,\ t\in(0,T), \tag{12}$$

with initial and ABC conditions given by

$$u(P_\ell,0)=\mathcal{U}_0(P_\ell),\qquad \frac{\partial u}{\partial t}(P_\ell,0)=\mathcal{W}_0(P_\ell),\qquad \ell\in\mathcal{I}, \tag{13}$$

$$\frac{1}{\sqrt{c_0}}\frac{\partial u}{\partial t}(P_\ell,t)+\frac{\partial u}{\partial \mathbf{n}}(P_\ell,t)=0,\qquad \ell\in\mathcal{I}_{\partial\Omega},\ t\in(0,T). \tag{14}$$

The semi-discrete collocation problem (12)-(14) consists of finding a vector $\mathbf{u}=\{u_\ell,\ \ell\in\mathcal{I}\}$, corresponding to elements $\{u_{ij},\ i,j=1,...,\nu\}$ that allows to write the IGA solution as

$$u(\mathbf{x},t)=\sum_{i=1}^{\nu}\sum_{j=1}^{\nu}u_{ij}R^p_{ij}\circ\mathbf{F}^{-1}(\mathbf{x},t), \tag{15}$$

according to (10) and (11). We introduce the IGA collocation matrices $[D_r]$ associated to r-th derivatives at collocation points, with $r=0,\ 1,\ 2$, where D_0, D_1 and D_2 account for the identity, $\frac{\partial}{\partial\mathbf{n}}$ and Δ operators, respectively. Then equations (12)-(14) can be expressed in matrix form as a system of second-order ordinary differential equations [28]:

$$\frac{\partial^2}{\partial t^2}[D_0\mathbf{u}(t)]_\ell-c_0[D_2\mathbf{u}(t)]_\ell=[\mathbf{f}(t)]_\ell,\ell\in\mathcal{I}_\Omega,[D_0\mathbf{u}(0)]_\ell=[\mathcal{U}_\mathbf{0}]_\ell,\frac{\partial}{\partial t}[D_0\mathbf{u}(0)]_\ell=[\mathcal{W}_\mathbf{0}]_\ell,\ell\in\mathcal{I}, \tag{16}$$

$$\frac{1}{\sqrt{c_0}}\frac{\partial u}{\partial t}[D_0\mathbf{u}(t)]_\ell + [D_1\mathbf{u}(t)]_\ell = 0, \qquad \ell \in \mathcal{I}_{\partial\Omega} \tag{17}$$

where $[\mathbf{w}]_\ell$ is the ℓ-th element of a vector $\mathbf{w}$ and $[D_r]_\ell$ is the ℓ-th row of the collocation matrix D_r, $r = 0, 1, 2$. Then, $\forall \ell \in \mathcal{I}$, $\mathbf{u}(t) := [u(P_\ell, t)]$, $\mathbf{f}(t) := [f(P_\ell, t)]$, $\mathcal{U}_\mathbf{0} := [\mathcal{U}_0(P_\ell)]$, $\mathcal{W}_\mathbf{0} := [\mathcal{W}_0(P_\ell)]$, and all elements are equal to zero elsewhere.

3.2 IGA Galerkin discretization of the acoustic problem

Starting from the variational form of the acoustic wave problem (4) we replace the L^2-inner products and the bilinear form (5) with their IGA quadrature-based approximations. Then the semidiscrete continuous-in-time problem reads: for each $t \in (0, T)$, find $u_h \in \mathcal{N}_h$ such that:

$$\left(\frac{\partial^2 u_h}{\partial t^2}, v\right)_h + a_h(u_h, v) + \sqrt{c_0} < \frac{\partial u_h}{\partial t}, v >_{h,\partial\Omega} = (f, v)_h \quad \forall v \in \mathcal{N}_h, \tag{18}$$

where $(\cdot,\cdot)_h$, $a_h(\cdot,\cdot)$ and $< \cdot,\cdot >_{h,\partial\Omega}$ are the IGA L^2- quadrature, stiffness and boundary bilinear forms, respectively. The algebraic form of (18) is obtained by representing discrete functions in the IGA basis functions, thus yielding a system of second-order ordinary differential equations:

$$\mathcal{M}\ddot{\mathbf{u}}(t) + \mathcal{C}\dot{\mathbf{u}}(t) + \mathcal{A}\mathbf{u}(t) = \mathbf{F}(t) \tag{19}$$

with initial conditions $\mathbf{u}(0) = \mathcal{U}_\mathbf{0},\ \dot{\mathbf{u}}(0) = \mathcal{W}_\mathbf{0}$ (see [27] for details). In system (19), $\mathcal{M}$ and $\mathcal{A}$ are the mass and stiffness IGA Galerkin matrices associated to $a_h(\cdot,\cdot)$ and $(\cdot,\cdot)_h$ respectively, whereas $\mathcal{C}$ accounts for the boundary term $\sqrt{c_0} < \cdot,\cdot >_{h,\partial\Omega}$. We recall that $\mathcal{M}$, $\mathcal{C}$ and $\mathcal{A}$ are symmetric, $\mathcal{M}$ is positive definite, whereas $\mathcal{A}$ is positive semi-definite, $\mathcal{C}$ is positive semi-definite with most elements equal to zero. Finally, $\forall t \in (0, T), \mathbf{u}(t)$ is the vector of coefficients of u_h in the IGA basis, and $\mathbf{F}(t)$ is a known vector accounting for the contribution of f.

3.3 The Newmark IGA collocation and Galerkin schemes

We partition the temporal interval $[0, T]$ into N subintervals $[t_{n-1}, t_n]$, with $t_0 = 0$, $t_N = T$, $\Delta t = T/N$, $t_n = n\Delta t$, $n = 1, ..., N$, and apply the Newmark method [20] to the numerical semi-discrete continuous-in-time approximation of the acoustic wave IGA collocation problem (16)-(17). We obtain the set of recurrence relations at collocation points:

$$[D_0]_\ell \frac{\mathbf{u}_{n+1} - 2\mathbf{u}_n + \mathbf{u}_{n-1}}{\Delta t^2} - c_0[D_2]_\ell\Big[\beta\mathbf{u}_{n+1} + \Big(\frac{1}{2} - 2\beta + \gamma\Big)\mathbf{u}_n + \Big(\frac{1}{2} + \beta - \gamma\Big)\mathbf{u}_{n-1}\Big] =$$

$$\Big[\beta\mathbf{f}_{n+1} + \Big(\frac{1}{2} - 2\beta + \gamma\Big)\mathbf{f}_n + \Big(\frac{1}{2} + \beta - \gamma\Big)\mathbf{f}_{n-1}\Big]_\ell, \quad \ell \in \mathcal{I}_\Omega \tag{20}$$

$$\frac{1}{\sqrt{c_0}}[D_0]_\ell \frac{\gamma\mathbf{u}_{n+1} + (1 - 2\gamma)\mathbf{u}_n + (\gamma - 1)\mathbf{u}_{n-1}}{\Delta t} + [D_1]_\ell\mathbf{u}_{n+1} = 0, \quad \ell \in \mathcal{I}_{\partial\Omega}. \tag{21}$$

We enforce the average of normal derivatives at any corner point.
If we consider now the IGA Galerkin approximation (19), the application of the Newmark scheme gives the recurrence relations:

$$\mathcal{M}\frac{\mathbf{u}_{n+1} - 2\mathbf{u}_n + \mathbf{u}_{n-1}}{\Delta t^2} + \mathcal{C}\frac{\gamma\mathbf{u}_{n+1} + (1 - 2\gamma)\mathbf{u}_n + (\gamma - 1)\mathbf{u}_{n-1}}{\Delta t} + \tag{22}$$

$$\mathcal{A}\Big[\beta\mathbf{u}_{n+1} + \Big(\frac{1}{2} - 2\beta + \gamma\Big)\mathbf{u}_n + \Big(\frac{1}{2} + \beta - \gamma\Big)\mathbf{u}_{n-1}\Big] = \Big[\beta\mathbf{F}_{n+1} + \Big(\frac{1}{2} - 2\beta + \gamma\Big)\mathbf{F}_n + \Big(\frac{1}{2} + \beta - \gamma\Big)\mathbf{F}_{n-1}\Big].$$

Remark 1. In both collocation and Galerkin versions, the second initial vector $\mathbf{u}_1$ can be computed from the first one $\mathbf{u}_0$ associated to initial condition (2)-left applying a second-order explicit one-step method, e.g., an explicit two-stage Runge-Kutta method, thus preserving the global accuracy of the numerical scheme with respect to the time step Δt, and using (2)-right.⧫

In spite of the fact that the theoretical analysis for IGA collocation discretizations of elliptic problems in two and three dimensions is still an open issue, several numerical studies investigate the convergence of the method with respect to the discretization parameters p, h and k, in various fields of application (e.g., [1, 16, 18, 24]). In this respect, in our previous works [27]-[29] we presented a detailed numerical study of stability, convergence and computational cost of the IGA collocation and Galerkin approximation of the acoustic wave equation with ABCs. In the present work, we focus on the comparison of spectral properties of IGA Collocation and Galerkin matrices arising from implicit Newmark schemes, which yield at each time step the solution of the linear system

$$\mathcal{K}\mathbf{u}_{n+1} = \mathbf{\Upsilon}(t_{n+1}, t_n, t_{n-1}), \tag{23}$$

with iteration matrix

$$\mathcal{K} = [D_1]_{\ell\in\mathcal{I}_{\partial\Omega}} + \frac{\gamma}{\Delta t\sqrt{c_0}}[D_0]_{\ell\in\mathcal{I}_{\partial\Omega}} - c_0\beta[D_2]_{\ell\in\mathcal{I}_\Omega} + \frac{1}{\Delta t^2}[D_0]_{\ell\in\mathcal{I}_\Omega}, \tag{24}$$

in the case of IGA collocation (20)-(21), and

$$\mathcal{K} = \frac{\gamma}{\Delta t}\mathcal{C} + \beta\mathcal{A} + \frac{1}{\Delta t^2}\mathcal{M}, \tag{25}$$

in the case of IGA Galerkin (22), respectively. Furthermore, the right terms $\mathbf{\Upsilon}(t_{n+1}, t_n, t_{n-1})$ account for the values of data functions $\mathcal{U}_0$, $\mathcal{W}_0$, f, at times t_{n+1}, t_n, t_{n-1}.

Remark 2. By using Taylor expansions it can be proven that the Newmark method is first-order accurate with respect to Δt if $\gamma \neq \frac{1}{2}$, and it is second-order if $\gamma = \frac{1}{2}$. The schemes (20)-(21) and (22) are considered explicit when $\beta = 0$, even if the matrices D_0 in (20) and $\mathcal{M}$ in (22) are not diagonal. Moreover, they coincide with the *Leap-Frog* method when $\gamma = \frac{1}{2}$, which in particular is explicit and second-order accurate with respect to Δt. Nevertheless, the IGA matrices associated to collocation (20)-(21) and Galerkin (22) approximations become denser for increasing p and k, both for explicit ($\beta = 0$) and implicit ($\beta \neq 0$) case, since the corresponding IGA mass matrices are not diagonal. ⧫

4 Condition number estimates and numerical results

In Section 3.3 we showed that each step of either the explicit or the implicit method involves the resolution of a linear system, that may be dense and ill-conditioned depending on the choice of the IGA parameters. In addition, the behavior of the spectral properties of IGA collocation and Galerkin matrices is of interest not only for possible investigation of efficient preconditioned iterative solutions of the linear systems arising at each step, but also in order to estimate the maximum allowable time step Δt for explicit Newmark schemes. Unfortunately, the theoretical analysis of spectral bounds for IGA matrices is still an open issue, since the results on eigenvalues and condition numbers of the IGA mass and stiffness matrices are almost conjectures. With regard to this, we recall some condition number estimates that are reported in [10] in the two dimensional case for the Galerkin isogeometric mass ($\mathcal{M}$) and stiffness ($\mathcal{A}$) matrices, for the approximation of the Poisson equation with Dirichlet boundary conditions. Regardless of the k-regularity of the spline basis functions, the following bounds are shown:

$$cond(\mathcal{M}) \le cp^2 16^p, \quad \text{with } c \text{ independent of } h \text{ and } p, \quad cond(\mathcal{A}) \le c(h)p^8 16^p. \tag{26}$$

In addition, some bounds on the extreme eigenvalues are proven in [10] in the one dimensional case, and in [11] also for the case of dimension $d > 1$. Furthermore, a methodical numerical study has been carried out in [12] in order to investigate the conditioning of Galerkin Isogeometric mass and stiffness matrices in d dimensions, yielding the following more detailed estimates:

$$\text{for } k = 0 \text{ regularity:} \qquad cond(\mathcal{M}) \approx p^{-d/2} 4^{pd}, \tag{27}$$

$$\text{for } k = p-1 \text{ regularity:} \qquad cond(\mathcal{M}) \approx e^{pd} \quad \text{if } h \le 1/p \tag{28}$$

$$\text{for } k = 0 \text{ regularity:} \qquad cond(\mathcal{A}) \approx \begin{cases} h^{-2}p^2 & \text{if } h \le (p^{2+d/2}d^{-dp})^{1/2} \\ p^{-d/2}4^{pd} & \text{otherwise,} \end{cases} \tag{29}$$

$$\text{for } k = p-1 \text{ regularity:} \qquad cond(\mathcal{A}) \approx \begin{cases} h^{-2}p & \text{if } h \le e^{-dp/2} \\ pe^{pd} & \text{if } e^{-dp/2} \le h \le 1/p. \end{cases} \tag{30}$$

In [30] we have conducted a systematic numerical study of the eigenvalue distribution, condition numbers and sparsity of the mass and iteration matrices arising from the IGA collocation approximation in space and Newmark advancing schemes in time, both explicit and implicit, of the acoustic wave equation in the reference square domain with Dirichlet, Neumann and absorbing boundary conditions.

We present now a numerical study of the condition numbers of the mass matrix $\mathcal{M}$ and iteration matrix $\mathcal{K}$, for the acoustic wave problem in the reference square domain $\Omega = [0,1] \times [0,1]$ and in the quarter of circular ring domain with external and internal radius of 2 and 1, respectively, varying the degree p, regularity k and mesh size h, focusing on the comparison of the IGA collocation method introduced in Section 3.1 and of the IGA Galerkin method introduced in Section 3.2. We fix the time discretization parameters of the Newmark scheme $\Delta t = 0.1$, $\beta = 0.5$, $\gamma = 0.5$, corresponding to a second order accurate implicit scheme. All tests have been carried out in 2D with MATLAB R2024b using the GeoPDEs 3.0 library (e.g. [7, 26]). In particular, the collocation matrices introduced in Section 3.1 are built using the structure `sp_eval`, whereas the condition numbers are computed as the ratio $(max|\lambda|)/(min|\lambda|)$ between extreme eigenvalues of a given matrix.

Condition number of the mass matrix. In fig. 1 we report the condition numbers $\mathsf{cond}(\mathcal{M})$ of the IGA collocation ($\square$ symbols) and Galerkin ($\circ$ symbols) mass matrices for the acoustic wave problem, versus h, for $p = 3$ (top), and versus p, for $h = 1/16$ (bottom), for minimum regularity $k = 1$ (left) and maximum regularity $k = p - 1$ (right). Dashed lines refer to square domain, whereas dotted lines refer to quarter-of-ring domain. The numerical results show that if p is fixed (top), the condition numbers $\mathsf{cond}(\mathcal{M})$ are almost always independent of h, in agreement with estimates (27)-(28). For the p- refinement with fixed h (bottom), the condition number $\mathsf{cond}(\mathcal{M})$ of the IGA Galerkin mass matrix grows as $p^{-1}4^{2p}$ in the case of minimal regularity $k = 1$, whereas for maximal regularity $k = p - 1$ the growth is e^{2p}. In the case of IGA collocation the numerical results are better than the ones predicted in estimates (27)-(28), since the condition number $\mathsf{cond}(\mathcal{M})$ grows as $p^{-1}4^{\frac{3}{2}p}$ in the case of minimal regularity $k = 1$ and e^p for maximal regularity $k = p - 1$.

In fig. 2, we report the condition numbers $\mathsf{cond}(\mathcal{M})$ for increasing k and four values of polynomial degree p, fixed $h = 1/16$. The left (resp. right) panels refer to square (resp. quarter-of-ring)

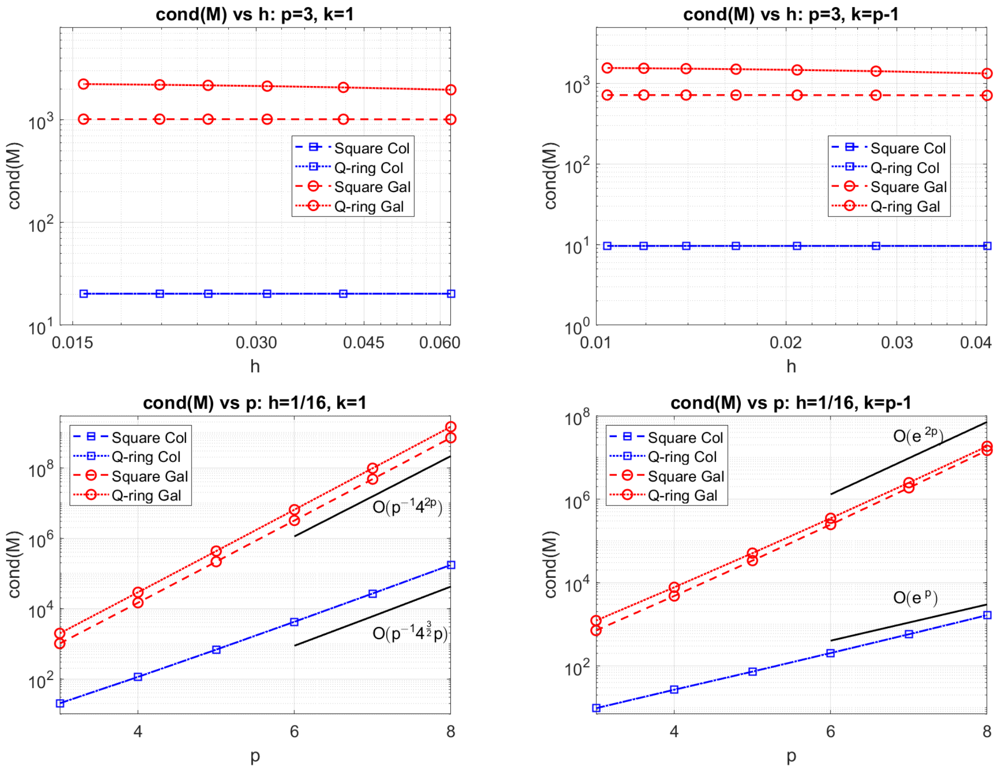

Figure 1: Condition number $\mathsf{cond}(\mathcal{M})$ of the mass matrix. Left: $k = 1$; right $k = p-1$. Top: vs h, fixed $p = 3$; bottom: vs p, fixed $h = 1/16$. Dashed (resp. dotted) lines refer to square (resp. quarter-of-ring) domain. The □ (resp. ○) symbols refer to IGA collocation (resp. Galerkin).

domain; the top (resp. bottom) panels refer to IGA collocation (resp. Galerkin) approximation. It seems that $\mathsf{cond}(\mathcal{M})$ decreases exponentially when the regularity k increases.
We observe that all above considerations are valid regardless of the shape of the domain, either square or quarter-of-ring, and, in addition, the values of $\mathsf{cond}(\mathcal{M})$ for the IGA Galerkin case are always higher than for the collocation case.
Condition number of the iteration matrix with absorbing boundary conditions. In fig. 3 we report the condition numbers $\mathsf{cond}(\mathcal{K})$ of the IGA collocation and Galerkin iteration matrices for the acoustic wave problem with absorbing boundary conditions, versus h, for $p = 3$ (top), and versus p, for $h = 1/16$ (bottom), using the same setting as in fig. 1. The numerical results show that if p is fixed and h is suitably small, the condition numbers $\mathsf{cond}(\mathcal{K})$ seem to grow no more as h^{-2}, whereas they are almost always independent of h when h increases, in agreement with estimates (29)-(30). For the p- refinement with fixed h, regardless of square or quarter-of-ring domain, the condition number $\mathsf{cond}(\mathcal{K})$ of the IGA Galerkin mass matrix grows as $p^{-1}4^{2p}$ both in the case of minimal regularity $k = 1$ and maximal $k = p - 1$. Similarly to what we have observed for the IGA mass matrices, the numerical results are better in the case of IGA collocation iteration matrices, since the condition number $\mathsf{cond}(\mathcal{K})$ grows no higher as $p^{-1}4^p$ both in the case of minimal regularity $k = 1$ and maximal $k = p - 1$, improving again estimates (29)-(30).
In fig. 4, we report the condition numbers $\mathsf{cond}(\mathcal{K})$ for increasing k and four values of polynomial

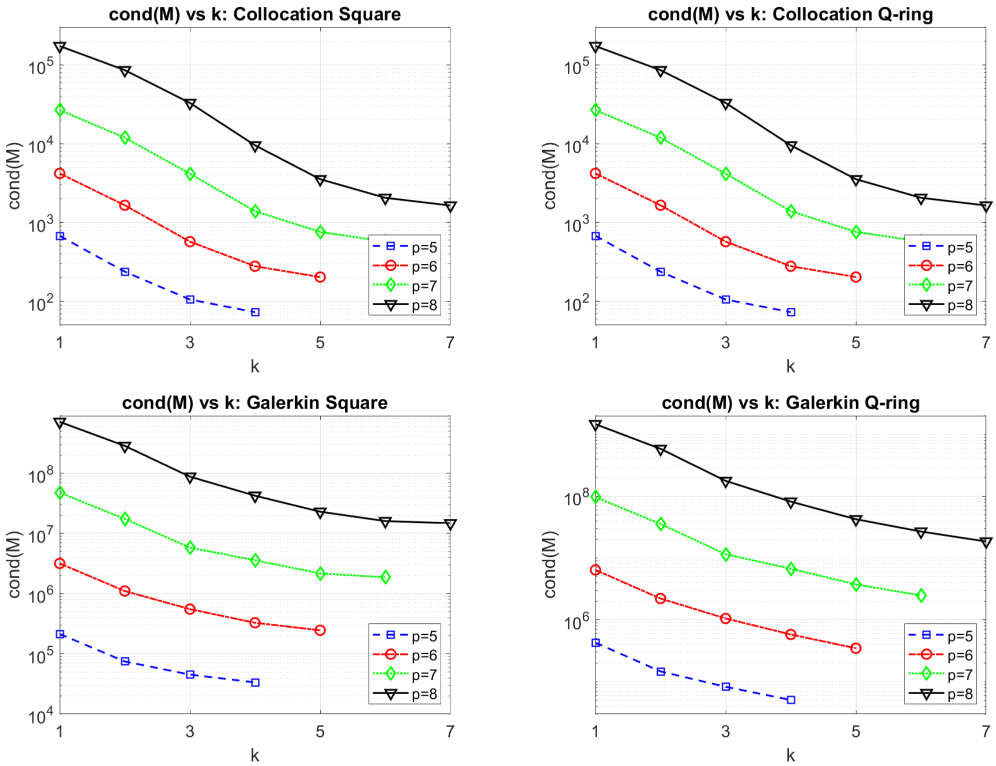

Figure 2: Condition number $\mathsf{cond}(\mathcal{M})$ of the mass matrix vs k, for $p = 5,\ 6,\ 7,\ 8$, fixed $h = 1/16$. Left: square domain; right: quarter-of-ring domain. Top: IGA collocation; bottom: IGA Galerkin.

degree p, fixed $h = 1/16$, and the same setting of plots as in fig. 2. It seems that $\mathsf{cond}(\mathcal{K})$ decreases exponentially when the regularity k increases, but it increases when the regularity k is close to the maximum value $p-1$. As before, we note that all above considerations are valid regardless of the shape of the domain, and, in addition, the values of $\mathsf{cond}(\mathcal{K})$ for the IGA Galerkin case are always higher than for the collocation case.

5 CONCLUSIONS

In this paper we have studied experimentally the condition number of the mass and iteration matrices related to the IGA collocation and Galerkin approximation of the acoustic wave equation with first order absorbing boundary conditions, while the time-advancing scheme is based on Newmark method. This analysis is of interest not only for stability analysis of explicit Newmark schemes in order to estimate the maximum allowable time step Δt, but also for the investigation of efficient preconditioned iterative solutions of the linear systems arising at each step of the Newmark schemes. We have presented a direct numerical comparison of the collocation and Galerkin IGA methods with regard to the condition numbers of their mass and iteration matrices, varying the polynomial degree p, mesh size h and regularity k, with reference also to the theory and conjectures available for matrices resulting from IGA Galerkin approximation of the Laplacian with Dirichlet boundary conditions. Our results show that similar bounds hold for the condition numbers of mass and iteration matrices for IGA discretizations of acoustic

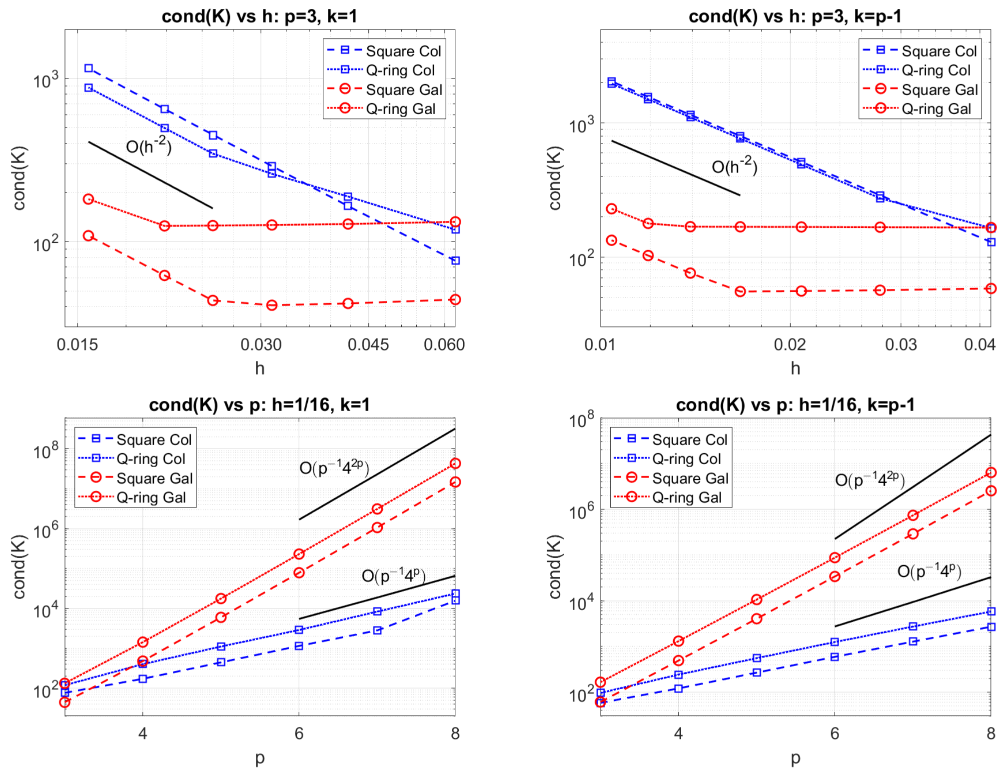

Figure 3: Condition number $\mathsf{cond}(\mathcal{K})$ of the iteration matrix. Left: $k = 1$; right $k = p - 1$. Top: vs h, fixed $p = 3$; bottom: vs p, fixed $h = 1/16$. Dashed (resp. dotted) lines refer to square (resp. quarter of ring) domain. The $\square$ (resp. $\circ$) symbols refer to IGA collocation (resp. Galerkin).

wave problems, and, in most cases, the collocation bounds are better than the Galerkin ones.

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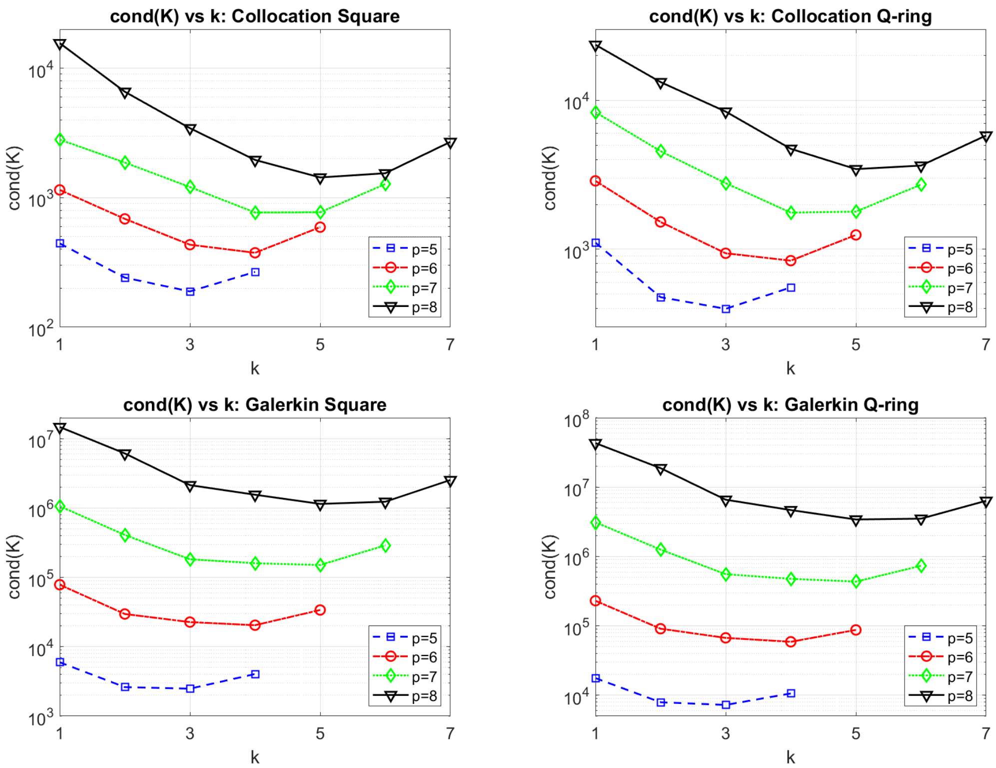

Figure 4: Condition number $\mathsf{cond}(\mathcal{K})$ of the iteration matrix vs k, $p = 5,\ 6,\ 7,\ 8$. Left: square domain; right: quarter of ring domain. Top: IGA collocation; bottom: IGA Galerkin.

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