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AN EQUILIBRIUM FINITE ELEMENT FORMULATION FOR REINFORCED CONCRETE

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Summary. This paper presents a 3D equilibrium finite element approach for reinforced concrete, based on a piecewise linear tetrahedral elements for concrete stress interpolation and embedded rebars represented as curves within the volume mesh. Global equilibrium is enforced through traction continuity equations on each triangular face of the mesh, incorporating the rebar stress contribution on faces intersected by rebars. The formulation extends to the elastoplastic case and limit analysis by adding nonlinear constraints: semidefinite constraints for concrete stress based on the Rankine criterion, while the material behavior of the rebars are modeled using a 1D perfect elastoplastic model.

1 INTRODUCTION

The development of equilibrium finite element (EFE) approaches dates back at least to the work of Fraeijs de Veubeke in 1965 [4]. Since then, EFEs have been extensively researched [1, 2, 3, 5, 6, 7, 8], but they have not achieved the same success in structural analysis as their dual counterpart, the displacement-based approach. To our knowledge, very few commercial software packages incorporate EFE in their finite element libraries. This is despite their advantage of providing a stress distribution that locally satisfies the equilibrium equations, which is crucial for verifying the resistance of different parts of a structure.

Another advantage of the equilibrium approach is that, when coupled with a displacement approach, it provides an error estimator that can be used for mesh refinement. This combination also helps validate the numerical results used in various post-processing steps aimed at checking the resistance and ensuring the validity of the structure design. Thus, the equilibrium approach is a powerful tool for structural analysis that should be more widely adopted and incorporated into commercial software packages.

To advance this objective, a novel equilibrium finite element formulation for reinforced concrete was introduced in [5]. Understanding the behavior of reinforced concrete structures presents a formidable challenge due to the intricate interplay between ductile reinforcements and brittle concrete. Unlike steel, where the normality flow rule aligns with reality [3, 2], reinforced concrete does not readily conform to a perfectly plastic model, hindering the application of robust mathematical frameworks for nonlinear analysis. Nonetheless, through extensive examination of various test cases, it has been demonstrated that the equilibrium formulation offers a promising tool for the analysis of concrete structures and can also be employed in limit analysis to predict the load-bearing capacity of these structures.

In the first section of this paper, we will present the equilibrium formulation for reinforced concrete in the elastic case. This formulation will be extended to the elastoplastic case and limit analysis in subsequent sections. We will conclude with examples that illustrate the efficiency of the proposed formulation.

2 THE EQUILIBRIUM FINITE ELEMENT FORMULATION:

2.1 Brief overview without rebars:

The equilibrium formulation is based on the following equations:

$$\operatorname{div} \boldsymbol{\sigma} = \mathbf{0} \,. \tag{1}$$

In order to verify strongly (1) in the volume V of the solid, the stress tensor $\boldsymbol{\sigma}$ needs to be at least continuous and differentiable, a condition that is not easy to ensure in the general 3D case. An alternative approach is to consider a partition of the volume $V = \bigcup_{i=1}^{n_{vol}} V_i$, where we apply the following transformation of the equilibrium equations :

$$\operatorname{div} \boldsymbol{\sigma} = \mathbf{0} \quad \text{on } V \implies \begin{cases} \operatorname{div} \boldsymbol{\sigma}_i = \mathbf{0} & \text{on all } V_i \\ [\![\boldsymbol{\sigma}]\!]_i \, \boldsymbol{n} = \mathbf{0} & \text{on all } \Delta_i \end{cases} , \tag{2}$$

where $\boldsymbol{\sigma}_i$ corresponds to the concrete stress tensor of the i^{th} elementary volume V_i, the notation $[\![\boldsymbol{\sigma}]\!]_i = \boldsymbol{\sigma}_j - \boldsymbol{\sigma}_k$ corresponds to the stress gap at an interface Δ_i between the two elementary volumes j and k, and $\boldsymbol{n}$ the normal vector to Δ_i.

From the transformed equilibrium equations, it is straightforward to obtain the following integral form of the equilibrium equations:

$$\begin{cases} \operatorname{div} \boldsymbol{\sigma}_i = \mathbf{0} & \text{on all } V_i \\ [\![\boldsymbol{\sigma}]\!]_i \, \boldsymbol{n} = \mathbf{0} & \text{on all } \Delta_i \end{cases} \implies \forall \bar{V} \in V, \quad \int_{\bar{V}} \operatorname{div} \boldsymbol{\sigma} \, dV = \mathbf{0} \,, \tag{3}$$

that can be considered as the true fundamental form of the equilibrium equations, as it does not require a C^1 stress tensor over the domain V.

Using a piecewise linear interpolation for the concrete stress tensor, it is straightforward to ensure $\operatorname{div} \boldsymbol{\sigma} = \mathbf{0}$ for each elementary volume V_i. By adding the traction continuity equations as equality constraints on each interface of the partition, the fundamental form of the equilibrium equations in (3) is then verified.

The equilibrium approach for the elastic case can be written as an optimization problem, expressed in the following form:

$$
\begin{cases}
\min_{\boldsymbol{\sigma}\in S^3} \sum_{i=1}^{n_{vol}} \frac{1}{2}\int_{V_i} \boldsymbol{\sigma} : \boldsymbol{C}^{-1} : \boldsymbol{\sigma}\, dV - \int_{\Gamma_u} \boldsymbol{\sigma n}\cdot \bar{\boldsymbol{u}}\, d\Gamma\,, & \text{(4a)}\\
\text{s.t.} \quad \operatorname{div}\boldsymbol{\sigma} = \boldsymbol{0} \quad \text{on all } V_i \in V\,, & \text{(4b)}\\
\qquad [\![\boldsymbol{\sigma}]\!]\,\boldsymbol{n} = \boldsymbol{0} \quad \text{on all } \Delta_i \in \Gamma_\Delta\,, & \text{(4c)}\\
\qquad \boldsymbol{\sigma}\,\boldsymbol{n} = \boldsymbol{t} \quad \text{on } \Gamma_t\,, & \text{(4d)}
\end{cases}
$$

where $\Gamma_u(\bar{\boldsymbol{u}})$ and $\Gamma_t(\boldsymbol{t})$ represent the boundary conditions for displacements and surface forces, respectively, $\Gamma_\Delta = \bigcup_{i=1}^{n_{face}} \Delta_i$ the set of all the interfaces of the volume partition, and $\boldsymbol{C}^{-1}$ the inverse of the elasticity tensor.

Extended the approach to the elastoplastic case can be performed by simply adding nonlinear material constraints on the concrete stress tensor. For concrete, one of the most widely used material criterion is the well-known Rankine criterion, expressed in as follows:

$$
-f_c \le \sigma_{III} \le \sigma_{II} \le \sigma_I \le f_t \quad \Rightarrow \quad \begin{cases} \boldsymbol{0} \preceq \boldsymbol{\sigma} + f_c \boldsymbol{I}_3 \\ \boldsymbol{0} \preceq -\boldsymbol{\sigma} + f_t \boldsymbol{I}_3 \end{cases}\,, \tag{5}
$$

where f_c is the compression limit of the concrete, f_t its tensile limit, $(\sigma_I, \sigma_{II}, \sigma_{III})$ the principal values of the stress tensor, and the notation $\boldsymbol{0} \preceq \cdot$ designates a positive semi-definite matrix.

The optimization problem expressed in (4), with the additional constraints in (5), is a semi-definite programming (SDP) optimization problem, where some of the unknowns are matrices that must be positive semi-definite. These optimization problem can be effectively solved by using the interior point algorithm.

2.2 The equilibrium approach with rebars:

For reinforced concrete, the rebars are considered as 1D curves $\boldsymbol{c}(s)$ embedded in the mesh. The discretization of each rebar is determined by the intersection points of its curve with the interfaces in the set Γ_Δ (see Fig. 1). Consequently, the rebar is divided into n_{rebar} segments, each with a piecewise constant rebar stress σ^r. Thus, for each rebar segment within an elementary volume, the equilibrium equation $\dfrac{d\sigma^r}{ds}$ is satisfied.

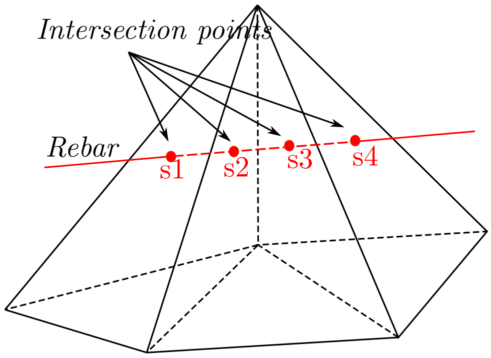


Figure 1: A rebar intersecting the interfaces of three tetrahedra.

Now, we introduce the key idea to connect the concrete stress tensor with the rebar stress in the equilibrium approach. To this end, the traction continuity constraint for an interface Δ_i intersected by a rebar is expressed in its integral form with the aid of following Lagrangian :

$$\mathcal{L}(\boldsymbol{\sigma}, \sigma^r, \boldsymbol{u}, u^r) = \int_{\Delta_i} [\![\boldsymbol{\sigma}]\!]\boldsymbol{n} \cdot \boldsymbol{u}\, d\Delta + S^r [\![\sigma^r]\!]\, u^r\,, \tag{6}$$

where S^r is the cross-section of the rebar, and $(\boldsymbol{u}, u^r)$ the Lagrange multipliers that can be assimilated to incompatible displacements on the interface Δ_i.

To connect the stress tensor $\boldsymbol{\sigma}$ and the rebar stress σ^r, the hypothesis of perfect bonding between $\boldsymbol{u}$ and u^r is adopted:

$$u^r(s_i) = \boldsymbol{e}^r(s_i) \cdot \boldsymbol{u}(s_i)\,, \quad \forall s_i \in I^r\,, \tag{7}$$

where $\boldsymbol{e}^r := d\boldsymbol{c}/ds$ is the tangent vector to the curve $\boldsymbol{c}(s)$ representing the rebar.

Replacing in (6) with the relation in (7), and linearizing the Lagrangian with respect to $(\boldsymbol{u}, u^r)$, we obtain the following variational equation:

$$\forall \delta\boldsymbol{u}\,, \quad \int_{\Delta_i} [\![\boldsymbol{\sigma}]\!]\boldsymbol{n} \cdot \delta\boldsymbol{u}\, d\Delta + S^r [\![\sigma^r]\!]\, \boldsymbol{e}^r \cdot \delta\boldsymbol{u}(s_k) = 0\,, \tag{8}$$

which represents the traction continuity equations at the interface Δ_i, coupling the concrete stress tensor $\boldsymbol{\sigma}$ and the rebar stress σ^r.

The Lagrange multipliers, i.e., the incompatible displacement vectors $\boldsymbol{u}$, are interpolated over the interface Δ_i: $\boldsymbol{u} = \boldsymbol{N}^T \boldsymbol{u}_N$. Thus, the equation (8) is re-written in the following form:

$$\forall \delta\boldsymbol{u}_N\,, \quad \delta\boldsymbol{u}_N^T \left(\int_{\Delta_i} \boldsymbol{N}[\![\boldsymbol{\sigma}]\!]\boldsymbol{n}\, d\Delta + S^r [\![\sigma^r]\!] \boldsymbol{N}(s_k)\boldsymbol{e}^r \right) = 0\,, \tag{9}$$

$$\Rightarrow \quad \int_{\Delta_i} \boldsymbol{N}[\![\boldsymbol{\sigma}]\!]\boldsymbol{n}\, d\Delta + S^r [\![\sigma^r]\!] \boldsymbol{N}(s_k)\boldsymbol{e}^r = \boldsymbol{0}\,, \tag{10}$$

where $\boldsymbol{u}_N$ is the nodal displacement vector at the interface vertices, $\boldsymbol{N}$ is the interpolation fuctions matrix.

We can now write the optimization problem for the equilibrium approach including rebars:

$$
\begin{cases}
\min\limits_{\boldsymbol{\sigma}\in S^3,\sigma^r\in\mathbb{R}} \displaystyle\sum_{i=1}^{n_{vol}} \frac{1}{2}\int_{V_i} \boldsymbol{\sigma} : \boldsymbol{C}^{-1} : \boldsymbol{\sigma}\, dV + \sum_{i=1}^{n_{rebar}} \frac{1}{2}\int_{\Delta L_i} \frac{(\sigma^r)^2}{E^r}\, ds - \int_{\Gamma_u} \boldsymbol{\sigma n}\cdot\bar{\boldsymbol{u}}\, d\Gamma\,, & \text{(11a)}\\
\quad \text{s.t.} \quad \operatorname{div}\boldsymbol{\sigma} = \boldsymbol{0} \quad \text{on } V_i\,, & \text{(11b)}\\
\qquad \boldsymbol{\sigma}\,\boldsymbol{n} = \boldsymbol{t} \quad \text{on } \Gamma_t\,, & \text{(11c)}\\
\qquad \displaystyle\int_{\Delta_i} \boldsymbol{N}[\![\boldsymbol{\sigma}]\!]\boldsymbol{n}\, d\Delta = \boldsymbol{0} \quad \text{on } \Delta_i \in \Gamma_\Delta \setminus \Gamma^r_\Delta\,, & \text{(11d)}\\
\qquad \displaystyle\int_{\Delta_i} \boldsymbol{N}[\![\boldsymbol{\sigma}]\!]\boldsymbol{n}\, d\Delta + S^r[\![\sigma^r]\!]\boldsymbol{N}(s_k)\boldsymbol{e}^r = \boldsymbol{0} \quad \text{on } \Delta_i \in \Gamma^r_\Delta\,, & \text{(11e)}\\
\qquad -f_c\boldsymbol{I}_3 \preceq \boldsymbol{\sigma} \preceq f_t\boldsymbol{I}_3 \quad \text{on } V_i\,, & \text{(11f)}\\
\qquad |\sigma^r| \leq f_y \quad \text{on } L^r\,, & \text{(11g)}
\end{cases}
$$

where Γ^r_Δ is the set of interfaces that are intersected by at least one rebar, and f_y is the elastic limit of the rebar.

2.3 The limit analysis problem for reinforced-concrete:

Limit analysis is a method that aims to directly compute the limit load of a structure in a fixed loading direction by solely considering the material's plasticity. The method is suited for sufficiently ductile structures, not limited by a deformation or a strain limit.

The optimization problem for limit analysis can be obtained by introducing a new variable λ into the problem in (11), representing the load factor to maximize, and by modifying the optimization function. The new optimization problem for limit analysis is then expressed in the following form:

$$
\begin{cases}
\max\limits_{\boldsymbol{\sigma}\in S^3,\sigma^r\in\mathbb{R},\lambda\geq 0} \lambda\,, & \text{(12a)}\\
\text{s.t.}\quad \operatorname{div}\boldsymbol{\sigma} = \boldsymbol{0} \quad \text{on } V_i\,, & \text{(12b)}\\
\boldsymbol{\sigma n} = \lambda \boldsymbol{t} \quad \text{on } \Gamma_t\,, & \text{(12c)}\\
\displaystyle\int_{\Delta_i} \boldsymbol{N}[\![\boldsymbol{\sigma}]\!]\boldsymbol{n}\, d\Delta = \boldsymbol{0} \quad \text{on } \Delta_i \in \Gamma_\Delta \setminus \Gamma_\Delta^r\,, & \text{(12d)}\\
\displaystyle\int_{\Delta_i} \boldsymbol{N}[\![\boldsymbol{\sigma}]\!]\boldsymbol{n}\, d\Delta + S^r[\![\sigma^r]\!]\boldsymbol{N}(s_k)\boldsymbol{e}^r = \boldsymbol{0} \quad \text{on } \Delta_i \in \Gamma_\Delta^r\,, & \text{(12e)}\\
-f_c\boldsymbol{I}_3 \preceq \boldsymbol{\sigma} \preceq f_t\boldsymbol{I}_3 \quad \text{on } V_i\,, & \text{(12f)}\\
|\sigma^r| \leq f_y \quad \text{on } L^r\,, & \text{(12g)}
\end{cases}
$$

The optimization problem is also an SDP problem, solved by using the interior point algorithm.

3 Examples:

3.1 Example 1: pure bending of a cantilever rectangular beam

To illustrate the efficiency of the proposed equilibrium formulation for reinforced concrete, we begin with a simple example of a cantilever rectangular beam. The beam has a length of 3 meters and cross-section dimensions of 1 meter by 0.5 meters, with a bending moment of 1 MN.m at its free end (see Fig. 2). It is reinforced with 3 layers of 3 rebars, each with a diameter of 25 mm. The concrete characteristics are a compression limit $f_c = 10$ MPa and a tensile limit $f_t = 0.01$ MPa. The elastic limit of the rebars is $f_y = \frac{500}{1.15} = 435$ MPa. The mesh of the beam is constituted of 22592 linear tetrahedra 3, while the mesh of the rebars will be determined by their intersection with mesh triangular interfaces.

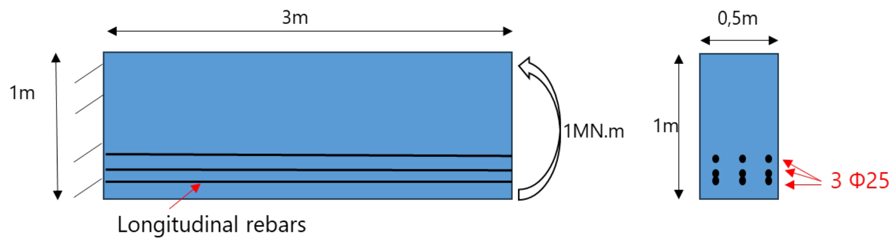


Figure 2: The geometry and boundary conditions of the cantilever beam.

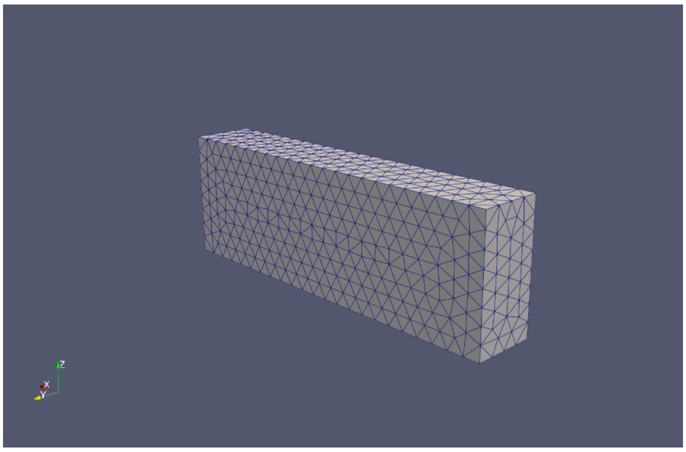
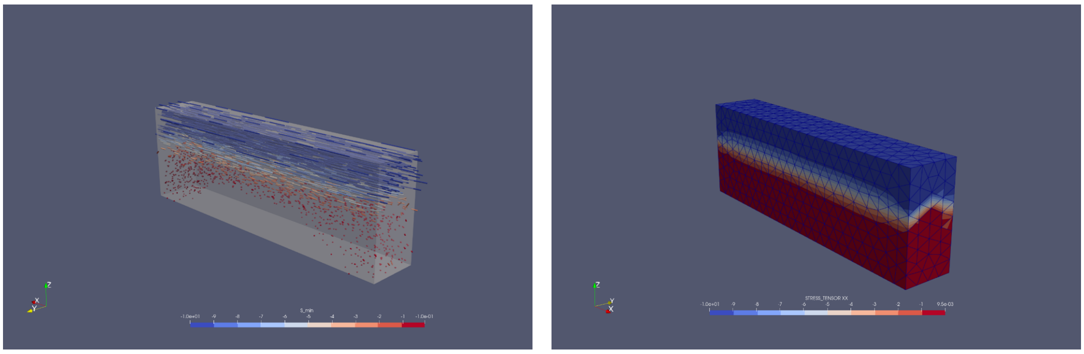


Figure 3: View of the mesh. 22592 linear tetrahedra.

For this example, we solve the limit analysis problem to determine the maximum loading factor for the applied bending moment. The results are presented in the figures below 4(a), 4(b) and 5, illustrating that the rebars are working at their elastic limit of 435 MPa and the concrete at its compression limit of 10 MPa. The solution of the optimization problem in (12) results in a loading factor of 1.32.



(a) Compression stress flow.

(b) The axial stress component σ_{xx}.

Figure 4: Concrete stress results.

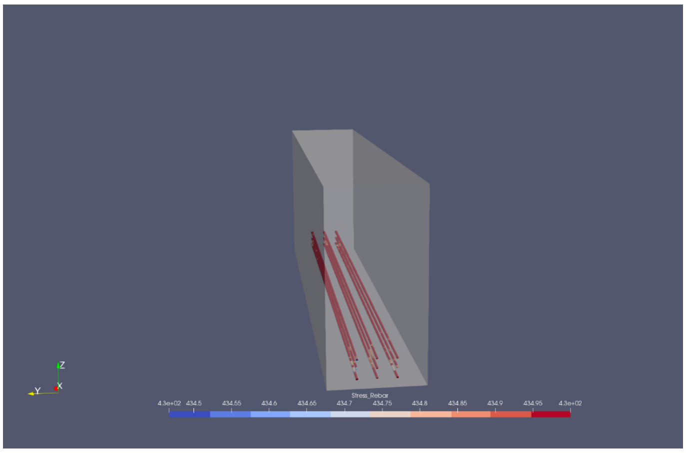


Figure 5: Rebars stress at their elastic limit 435MPa.

3.2 Example 2: rebar stress transmission through overlapping

In this second example, we consider the same dimensions and material characteristics of the rectangular cantilever beam. We utilize the same mesh as presented in Figure 3. The new arrangement of the rebars for this example is illustrated in Figure 6.

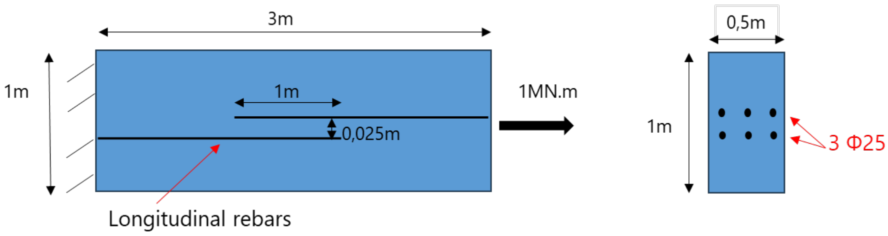


Figure 6: The geometry and boundary conditions of the cantilever beam.

In this example, we apply a limit analysis with a traction force applied at the free end of the beam. The objective of this case study is to verify and validate the capacity of the equilibrium approach to transmit axial stress from overlapping rebars, which is crucial for general applications in reinforced concrete structures.

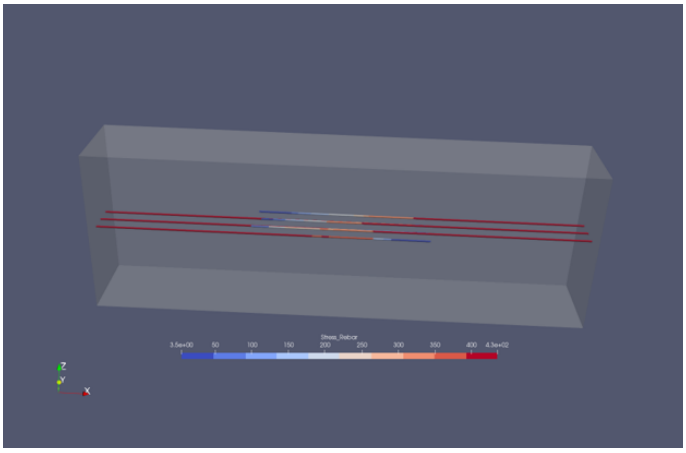


Figure 7: Rebars stress at their elastic limit 435MPa.

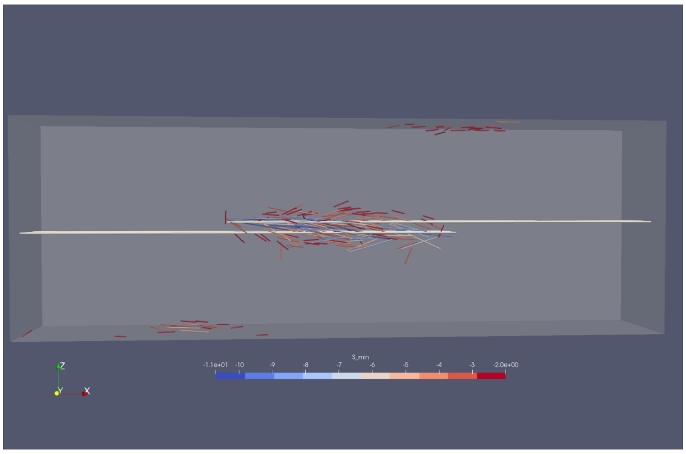


Figure 8: Rebars stress at their elastic limit 435MPa.

4 CONCLUSION

In conclusion, the equilibrium finite element approach has demonstrated significant potential in structural analysis despite its limited adoption compared to displacement-based methods. The inherent advantage of the equilibrium approach lies in its ability to provide a stress distribution that locally satisfies the equilibrium equations, crucial for structural resistance verification. Additionally, the synergy between the equilibrium and displacement approaches offers an efficient error estimator for mesh refinement and validates numerical results, enhancing the reliability of structural design.

This paper introduces a novel equilibrium formulation for reinforced concrete structures. Addressing the complexities arising from the combination of ductile reinforcements and brittle concrete, the equilibrium formulation has been shown to be a robust tool for analyzing reinforced concrete structures. Through the presented examples, we have demonstrated its efficiency for the elastoplastic analysis, as well as in limit analysis for predicting load-bearing capacities.

In future developments of the equilibrium approach, notable advancements include the use of limit analysis formalism for the automation of the strut-and-tie method in 3D structures [9], and the extension of the approach to accommodate large deformations/displacements case, further enhancing its applicability and robustness.

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