12th International Conference on Structural Analysis of Historical Constructions
SAHC 2020
P. Roca, L. Pelà and C. Molins (Eds.)

A COMPARISON BETWEEN TRADITIONAL AND MODERN APPROACHES FOR THE STRUCTURAL MODELLING OF BRICK MASONRY BARREL VAULTS

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Keywords: Historical Structure, Masonry, Modelling, Barrel Vaults

Abstract. *Masonry vaults are widespread and characteristic structural elements of our built heritage since many centuries, but for a very long time they were built only based upon the experience and the proportional analysis of previous positive examples. Since the Hooke's observations, in 17th century, about the shape of the catenary, and the first graphical analyses of 18th century, the tools for their "scientific" calculation have developed quickly [1], mainly to assess the stability of already existing structures rather than for the prevision of the future behaviour of new vaults. Despite the great progress in this field, ordinary programs for the static and seismic assessment of masonry buildings often disregard the vaults structural role and the professionals sometimes underestimate it, also due to the lack of attention dedicated to these structures by the technical codes. Therefore it seems now important to reconnect the elements of this modelling historical evolution, to compare the different methods and to find an equilibrium between complexity and reliability, making it accessible also to the common professional use, whose effects on preservation are important.*

To this aim, a pavilion vault was chosen as a reference, with given geometries and materials features, and the different methods were applied. On one side, traditional methods were chosen: the graphic Méry method [2] and the static theorem of limit analysis [3] have been applied to a system of 2D arches composing the vault. On the other side, a 2D Finite Element Model and the edge cutting ChronoEngine Distinct Element Model [4] have been also tested, under the same conditions. The influence of the brick pattern on the structural behaviour have been considered, conveniently defining the arches decomposition in the traditional methods and the blocks division in the Distinct Element Method. In all cases, calculations have been made changing both values and positions of the loads. The results are compared both in terms of stresses inside the masonry and in terms of deformation of the structural elements, evaluating the types of information and detail that the different approaches can supply. The results of the advanced numerical methods allow to assess the validity of the traditional approaches. On the other side, the possible contribution of the traditional methods to the calibration of the parameters for the numerical models is also discussed.

1 INTRODUCTION

Since the introduction of the first known arched masonry structures in Egypt and Mesopotamia, at least in the 4^{th} millennium b.C. [1], arches, vaults and domes have spread and characterized our historical built heritage with the adoption of various shapes and technologies, typical of each area and each historical period. As for all constructions – and for masonry structures in particular – until very recently also the curved masonry elements were built only based upon the experience and the proportional analysis of previous positive examples: the main concern of the builders was to have supports large enough to sustain the thrusts, rather than defining the thickness and shape of the vault itself. It was only in the 17^{th} century that Hooke made his observations about the shape of the catenary, concentrating the attention on the shape of the arch and its relation to the carried loads. Then, the first graphical analyses were developed in the 18^{th} century and since then the tools for the calculation of curved masonry structures in a scientific sense have developed quickly [2]. Despite the great progress in this field, and the advanced computation instruments available to date, most of the programs ordinarily adopted by the professionals for the static and seismic assessment of masonry buildings disregard the vaults structural role, involving an underestimation of their stability and a consequent overestimation of the needed structural interventions, in contrast with the required minimum intervention principle for historical buildings. Already in 1945 the architect and engineer Gustavo Giovannoni wrote that “the concept of limiting the strengthening interventions to the minimum needed entails the exploitation of the resource schemes developed in the building statics without altering them” [3], but to exploit these resources, which are intrinsic to the historical structures, we need to know them and also to quantify them efficiently: to this aim, a comparison among methods for the static analysis of barrel vaults with different, progressive levels of refinement are here reported to assess their reliability as instruments for the knowledge and preservation of these irreplaceable architectural elements.

2 STATE OF THE ART

Because of their geometry, barrel vaults are usually modelled as elemental parallel arches; of these, they inherit the different methods of analysis ([6], [26]). The methods for the assessment of arches and vaults can be roughly divided in two categories. To the first category includes the methods based on the graphical tracing of the thrust-line (i.e. the set of points corresponding to the position of the resultant of normal stresses in each cross section of the arch). To this category belong the well-known Mery’s method [5] and Heyman’s method [6]. The approach has also been extended from 2D to 3D by replacing the thrust-line with a thrust-network of forces ([19], [20], [21]) but for barrel vaults usually this is not necessary. The second category includes numerical approaches mainly based on finite element or discrete element methods. Each of them has pros and cons [2]. Because of the vastness of the subject matter, we will only describe the methods used in this paper.

2.1 Méry’s method

Mery's method [5], already reported in the 19th century treatises [22], is one of the simplest methods and perhaps for this reason it is widespread in design practice of small vaults. Mery assumes that a barrel vault can be studied as an arch made up of infinite non-tensile voussoirs that do not slide between each other. The method is based on the plotting of the line-of-thrust and the study of its position within the thickness of the arch. Based on experimental observations, it is assumed that in a round arch the line-of-thrust passes through the upper middle-third in the key section and at the lower middle-third in cross sections at 30° and 150° with respect to a horizontal line passing through the center of curvature. The arch is safe if the line-of-thrust does not cause tensile stresses, i.e. it is comprised in the cross-section core.

2.2 Line-of-thrust with minimum stress

Mery's method, not allowing the formation of cracks, is quite conservative. Other methods impose less strict conditions on the position of the line-of-thrust, usually adopting lower-bound theorem of limit analysis ([2],[6]). Among these, the approach implemented in the freeware software ARCO [10], computes the position of the line-of-thrust that minimizes the maximum stress in the sections. The stress, which is determined with a no-tension linear elastic model, is compared with the material strength to avoid compression failure. The approach has been used in this work because it is very common in Italian design practice and it adapts well to the stress checks required by the Italian code standard.

2.3 Finite Element Method

Vaults have been extensively studied with non-linear finite element models [2]. For instance, Creazza et al. [24] modelled a barrel vault in 3D using a nonlinear damage model. D’Altri et al. [17] modelled a gothic barrel vault using solid finite elements bricks interacting with contacts. Cavicchi and Gambarotta [25] proposed a two-dimensional finite element upper bound limit analysis that permits to model also the infill. If the infill is taken into account with equivalent forces, it is possible to model the arch with 2D nonlinear fiber beam elements, as proposed by de Felice [23]. This approach has been used in the present work because, despite its simplicity, it represents well the states of stress and the deformation of the arch.

2.4 Discrete Element Method

Discrete Element Method (DEM) has been widely used to model masonry vaults. By means of DEM, portions of vault or single bricks are modelled as blocks interacting each other by means of springs and dashpots. For a comprehensive review of the method, see [27]. Different authors used the software 3DEC to analyze the behavior of cross vaults [28][29][30] and barrel vaults [29] subjected to settlements of the imposts. In typical DEM the equations ruling the problem are solved using time-step explicit integration algorithms which are computationally intensive because of the small time-steps required. To reduce the computational effort, a particular type of DEM, called Non Smooth Contact Dynamics (NSCD), was proposed by Moreau and Jean [31][32]. In this case the equations ruling the problem are solved using implicit integration algorithms that permit larger time-step. This method was implemented in the software LMGC90 [33] and was used to study masonry arches and vaults [34][35][36][37]. An independent implementation of NSCD was proposed by Tasora et al. with the ProjectChrono open source library [7]. The library, originally developed for mechanical engineering, was applied to study masonry arches and domes [38][39]. The seismic behavior of local mechanisms in a castle were analyzed in [40] while Ferretti et al. [9] used the library to study masonry barrel vaults considering both the presence of linear elastic iron ties and filling, which is modelled with spheres.

Because of its features, the library Project Chrono was adopted in the present work.

3 CASE STUDY

In order to analyse and compare the potentialities, the effectiveness and also the limits of each of the aforementioned methods for the analyses of masonry barrel vaults, a case study was chosen from a real historical structure: the Pilotta Palace, in Parma (Italy). In particular, a barrel vault with pavilion heads, covering the atrium of the Farnese Palace, was chosen: the limited surveys and tests carried out allowed to hypothesize a structure made in brick masonry with a thickness varying between 12 and 24 cm (more details about the conformation of the analysed vaults can be found in [8]). Over the vault, a large room (8 m wide and 22 m long) is located, whose future uses are now under discussion, also in consideration of the structural capabilities of the vault itself [11]. The structural analyses started with the classical methods: the graphic statics of Mery's method and the more recent limit state analysis, conducted through the ARCO software. Since the deformability of the vault is not taken into account by these methods, the analysis was repeated using both Finite Elements Method (FEM) and Discrete Elements Method (DEM) in order to consider the consequences of geometry variations and the behavior of the tie and the supporting walls.

Figure 1: Analysed vault and overlying room

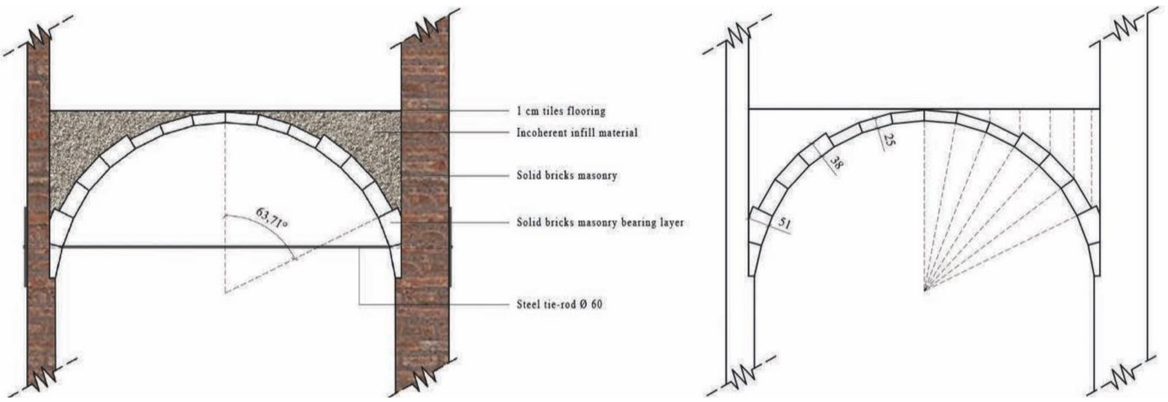


Figure 2: Structural scheme of the vault.

3.1 Méry's graphic method

The first tool used to analyse the vault is based upon the graphic statics: the Mèry's Method. Likewise other classical analysis models, it does not take in to account the deformability of the materials. A 1 m wide arch placed in a representative section of the considered vault, obtained by a detailed laserscanner architectural survey, was chosen to represent the whole structure. The specific weight for masonry and infill was assumed of 18 kN/m^3 and 14 kN/m^3 respectively. The analysis was performed with three load conditions: dead loads (g_k only), dead loads plus variable loads corresponding to reading room use category (q_k = 3 kN/m^2) and the same case with the introduction of the partial coefficients for the action's effect (γ) imposed by the Italian technical code in the Ultimate Limit States assessment. In particular, their values correspond to 1.0 for masonry and infill (γ_{G1}) and to 1.5 for flooring and variable loads (γ_{G2} and γ_Q). Moreover, the collapse condition was determined without partial coefficients, using an iterative procedure. The value of the mean compressive strength of masonry has been assumed equal to f_m = 3.2 MPa and the corresponding design compressive strength was f_d = 2.78 MPa (reduced to 0.93 MPa when taking into account the γ coefficients), following the Italian Code [4]. According to Mèry's method hypotheses, the round arch symmetrically loaded is studied with three plastic hinges located at 30°, 150° (haunches) and 90° (keystone) from the horizontal line passing through the centre of curvature. In fact, the only portion that was analyzed was the one comprised between 30° and 150° and it was divided into 12 voussoirs (Figure 2). The polygon of the forces, represented in Figure 3, is constrained to pass through the upper and lower limit of the cross section core at the keystone section and at the haunches sections respectively. Horizontal thrust component H values are shown in Table 1.

3.2 Line-of-thrust with minimum stress

The analysis was repeated using an approach based on the Line-of-thrust with minimum stress. The chosen tool was the software ARCO [10]: it varies iteratively the eccentricity of the line of thrust, aiming to achieve the lowest possible value of the stresses in the cross sections under the given scheme of vertical loads. The geometry was unchanged compared to the Mèry's method. Results are reported in Figure 3 and Table 1.

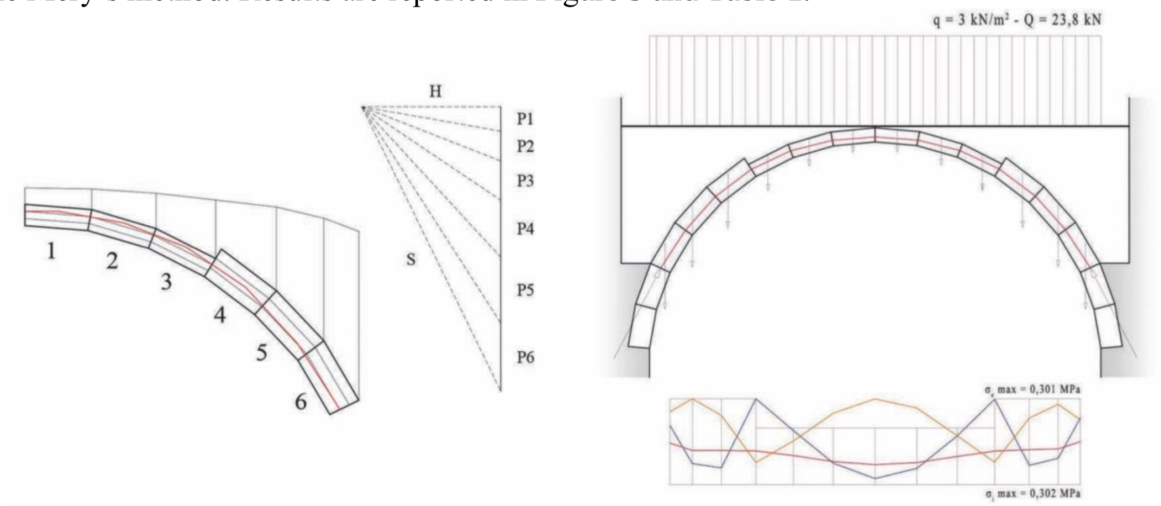
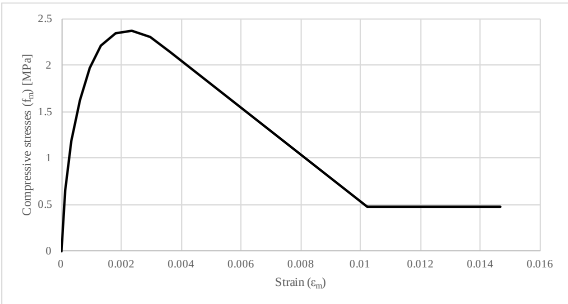


Figure 3: Analysis with Mèry's method (left) and ARCO software (right).

3.3 Finite Element Method

The tool used for the FEM analysis was the software Abaqus from Dassault Systèmes; the same geometry used in the classic methods was modeled with 2D fiber-beam finite elements. Dead and variable loads were rendered by concentrated forces applied in the centre of gravity of the elements. The collapse condition of the structure was obtained with the application of variable loads increased per-step. The mechanical behavior of masonry was modeled using the nonlinear material model "Concrete Damage Plasticity" (CDP), with Young modulus E = 1500 MPa and Poisson coefficient $\nu = 0.2$. The adopted CDP model parameters are: Dilatancy angle: 10°; Eccentricity: 0.1; f_{b0}/f_{c0}: 1.16; K: 0.667; Viscosity parameter: 0.002. For the stress-strain relationship in compression, the relationship proposed in [12] was used. Mortar's compressive strength was assumed prudently equal to f_m = 1 MPa, while the same value for the blocks was taken equal to f_b = 15 MPa. The obtained stress-strain curve (Figure 4) was rescaled to include plastic strain only. The tensile constitutive law refers to the experiences reported in [13] (Figure 5). The first batch of analyses was carried out with the vault constrained at the spring in order to better match the layout of the classic models. The values of the maximum compressive stress in the three cases are reported in Table 1, together with the horizontal thrust component H.



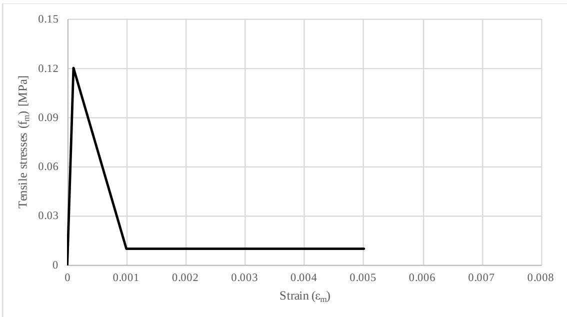


Figure 4 and 5: Compressive and tensile constitutive laws.

The second batch of analyses brings in the contributions of the supporting walls and of the metal tie-rods, 60 mm in diameter, both modelled with beam elements. Mechanical properties were deducted from previous studies regarding historical architectural tie-rods [14]. Considered values were Young modulus E_s = 130 GPa and maximum tensile strength R_m = 281 MPa; the corresponding strain is equal to ε_m = 6.5 %. This layout of the model takes into account only the situation of actual loads (dead loads), with both fixed anchors of the tie-rod and anchors penetrating into the masonry; this condition was simulated by a thermal elongation of the tie-rod calibrated on the value measured in the real case study. In this second case the horizontal thrust H decreases: this evidence means that due to the higher deformability of the tie, a portion of the forces at the haunches discharges on the supporting walls. When anchors penetration is present, plastic deformations develop at about 58° from the vault's axis (Figure 6), approximately in the section where the Mèry's method hypothesis expects the development of plastic hinges. In both cases, plasticization of the tie-rod was not observed. Horizontal displacements are concentrated on the left side of the structure (Figure 6), where supporting walls are thinner (Figure 1). Collapse occurs due to compressive failure of the masonry, with the tie-rod still working in elastic field. The maximum compressive stresses in the cross sections are 1.49 MPa and 0.42 MPa, depending on the fact that anchors penetration into the masonry was considered or not. This evidence highlights the connection between the deformability of the structure and its stress state under the same loading conditions.

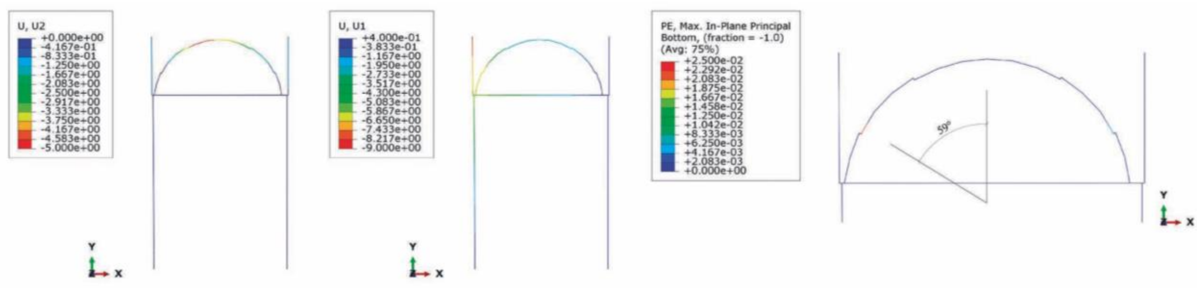


Figure 6: FEM analysis: vertical displacements, horizontal displacements and stresses.

3.4 Discrete Element Method

The analysis was then repeated with the DEM approach: the voussoirs were represented by macro-blocks with the same shape adopted in the previous methods, in order to preserve modelling homogeneity. Infill, flooring and variable loads were considered like concentrated forces applied in the centre of gravity of the elements, in order to reduce the computational burden, except for the case of collapse condition, in which infill was modelled with approximately 780 spheres. Real infill density was increased by a 1.5 factor in order to take into account the empty space among adjacent spheres. The geometry remains unchanged compared to previous analyses.

The software chosen for DEM analyses was ChronoEngine[7], an open source C++ library often used to study masonry architectural structures [9]. Contact forces among blocks and motion laws for each block are ruled by Coulomb’s friction law, Newton’s Second Law (in the gravitational field) and spring/dashpots reactions. A model based on springs and dashpots, if properly calibrated, can render mortar joints deformability and the resulting deformation of the vault. As a first attempt for calibration, values from [6] were adopted: damping was set at 0.2 kg/s, while compliance was 10-8 N/m. This second parameter was subsequently recalibrated for each different layout of the model, by iteratively matching displacements in a reference point obtained with a corresponding FEM model. The software does not directly provide values of compressive stresses.

Firstly, the 1 m wide arch representing the vault was modelled adopting fixed constrains at the spring, disregarding the contributions of the tie-rods and of the supporting walls. This configuration ensures a consistent comparison with the results obtained with classical methods (Table 1). The collapse condition, obtained using an increasing overload applied upon the flooring blocks in form of vertical concentrated loads, shows a very high lowering of the key section before the loss of bearing capacity of the structure.

The model was then improved with the introduction of the tie-rods and of the supporting walls (Figure 7). Walls were modelled as monolithic blocks, while the tie-rod was approximated with a linear-elastic spring, whose elastic properties are the same previously adopted in the FEM approach (the software does not allow an elasto-plastic behavior). The penetration of the anchors into the masonry was simulated thanks to a two-step analysis: the first one was conducted with a fictitious spring constant k’, calibrated with a similar FEM model, in order to obtain the desired elongation and the deformed configuration of the structure. In the second step, a tie-rod with the real spring constant k was added to the deformed vault obtained in the first step. When the infill is used, it produces a horizontal pressure that modifies the behavior of the model: an overturning mechanism of the upper wall is triggered (Figure 7), and the structure loses its bearing capacity even without the addition of increasing overloads. Displacement values computed in the various models configurations are shown in Table 2.

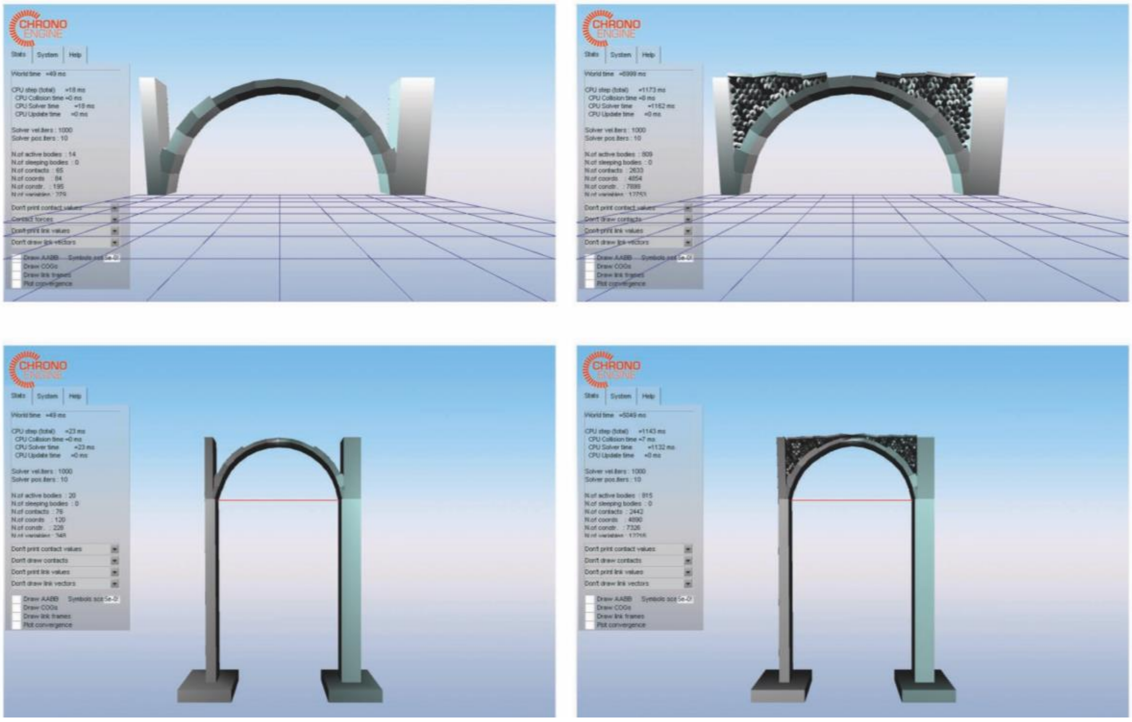


Figure 7: DEM analysis: vault constrained at the spring and model with tie-rod and supporting walls

4 RESULTS AND DISCUSSION

The results obtained with the four different methods are here discussed both in terms of forces and in terms of displacements (the latter, obviously, only for the numerical methods).

4.1 Horizontal thrust and compression stresses

Table 1 reports the horizontal thrust H and the compression stress obtained with the four different methods and with the four different load conditions (dead loads, dead loads plus live loads for reading room use, Ultimate Limit State, collapse condition). Their comparison allow to make some observations about the compliance of the different methods and about their reliability for a static safety assessment of the vaults.

The maximum values of compressive stresses in the first three load cases obtained with the Mery’s method are 0.486 MPa, 0.445 MPa and 0.546 MPa respectively. The value in collapse condition is equal to the design compressive strength value. The higher stress value in the case of dead loads only shows a positive impact of the variable loads on the vault stability. The results obtained with the ARCO software confirm the same trend observed in the previous approach; in fact the values for the maximum compressive stresses in the cross sections are 0.306 MPa, 0.301 MPa and 0.313 MPa for dead loads, reading room and Ultimate Limit States assessment respectively. Due to the optimization of the hinges position exploited by the software in order to minimize eccentricity values, those results are significantly lower with respect to those obtained from the Mèry's method. Nevertheless, the horizontal thrusts computed with ARCO are slightly larger than the ones obtained with Mery's method. FEM results in terms of thrust appear to be more similar to the Mery's ones, while the thrusts computed with the adopted DEM approach are larger also than the ARCO ones. The horizontal thrust at collapse are surprisingly close in Mery, FEM and DEM cases, while it is more than double if computed with ARCO, thus demonstrating that a widespread instrument adopted by professionals is prone to overestimate largely the thrust and thus possibly induces heavier interventions than really needed. From a static safety assessment point of view, in the case of the reading room live loads, the vault is verified with wide margins in the case of null or limited deformations (classical methods and numerical methods with constraints at the spring). However, when the deformability of the structure increases with the addition of supporting walls and tie-rods, this margin become thinner: the maximum overload that the structure can stand calculated with FEM results in 3.36 kN/m^2, only marginally bigger than the value of 3 kN/m^2 imposed by the Italian technical code for this intended use.

Table 1: Comparison of the results in terms of thrust and stress among the four adopted methods.

Method	Dead loads		Reading room		Limit state assessment		Collapse	
	H (kN)	Max comp. st. (MPa)	H (kN)	Max comp. st. (MPa)	H (kN)	max comp. st. (MPa)	H (kN)	max comp. st. (MPa)
Mèry	27.9	0.486	36.7	0.445	41.3	0.546	109.5	2.770
Safe theorem	30.4	0.306	40.4	0.301	45.8	0.313	278.4	2.780
FEM	31.7	0.385	40.7	0.391	45.4	0.394	102.6	1.326
FEM w. Sup.	29.5	0.421	-	-	-	-	-	-
FEM A. Ret.	24.5	1.493	-	-	-	-	35.5	1.495
DEM	38.1	-	45.2	-	50.1	-	104.3	-
DEM w. Sup.	67.9	-	-	-	-	-	-	-
DEM A. Ret.	15.7	-	-	-	-	-	-	-

4.2 Horizontal and vertical displacements

As already stated, only numerical methods produce results that take into account displacements and deformations. In Table 2, the horizontal displacements of the springing points Δh and the vertical displacements of the keystone are reported for each of the four load cases and for the two adopted numerical methods (FEM and DEM). Vertical deformations calculated with both FEM and DEM approaches are generally lower than the maximum observed value in the case study (estimated in about 90 mm): however, when the whole structure is taken into account (including anchors penetrating into the masonry) the expected drops in the keystone section are in the same order of magnitude. With regards to the DEM approach, occasional anomalies such as the very high keystone drop in collapse conditions without tie-rods and supporting walls could be explained with the difficulty in the calibration of the contact springs through the compliance parameter: in fact, it seems to mostly depend on the layout of the single model than on the overall mechanical properties of the masonry.

Table 2: Comparison of the results in terms of horizontal and vertical displacements among the 4 adopted methods.

Method	**Dead loads**		**Reading room**		**Limit state assessment**		**Collapse**	
	Δh (mm)	key lowering (mm)	Δh (mm)	key lowering (mm)	Δh (mm)	key lowering (mm)	Δh (mm)	key lowering (mm)
FEM	-	0.76	-	2.05	-	2.75	-	3.16
FEM w. Sup.	3.39	6.77	-	-	-	-	-	-
FEM A. Ret.	58.99	40.05	-	-	-	-	-	42.86
DEM	-	-1.03	-	0.62	-	1.36	-	680.68
DEM w. Sup.	7.02	39.20	-	-	-	-	-	-
DEM A. Ret.	58.40	53.40	-	-	-	-	-	-

12 CONCLUSIONS

Trying to reconnect the elements of the historical evolution of arches and vaults structural modelling, the present paper compares four different methods – two classical graphic ones and two modern numerical ones – looking for an equilibrium between complexity and reliability, for the common professional use. The main results can be summarized as follows:

- Good agreement in terms of thrust forces between graphical and numerical methods demonstrate that classical methods are trustworthy, altogether with a more straightforward application
- Classical methods do not take into account deformability, which is important when thin supporting walls or thin vaults are concerned: the results more consistent with the real observed ones were obtained with the finite elements method
- DEM method does not provide the state of stress in the blocks, therefore comparison with compressive strength of masonry is cumbersome and must be done in the post-processing phase.
- Calibration of the DEM is very delicate and case dependent, thus suitable mainly for edge cutting research case studies.

Acknowledgements. The contributions of Federica Ottoni to the analysis of the case study, of Claudio Boni to the calibration of the Discrete Element Method and of Alessandro Tasora to the ChronoEngine development are also gratefully acknowledged.

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