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PLASTIC DESIGN OF FRAME STRUCTURES UNDER UNCERTAIN CONDITIONS BY A STOCHASTIC MODEL

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Summary. This paper shows how probabilistic limit analysis of statically indeterminate frame structures can be done with the same simplicity as the deterministic limit analysis.

1 INTRODUCTION

The authors have developed FEM based limit and shakedown analysis of structural problems with uncertain data by stochastic optimization as an alternative to plastic reliability analysis [1]. They have developed chance-constrained programming with individual chance constraints for normally and lognormally distributed strength and loading to calculate limit and shakedown loads for prescribed reliability levels with the FEM [2,3].

Limit analysis is mainly used in civil engineering practice and teaching in the analysis and design of statically indeterminate truss and frame structures. Therefore the chance-constrained program with individual chance constraints is formulated here for frame structures with uncertain plastic moments to demonstrate the application of the concept to the standard problem of a portal frame. Truss girders are discussed in [4].

2 DIRECT METHODS FOR PLASTIC ANALYSIS OF FRAME STRUCTURES

Consider a retangular portal frame made of perfectly plastic elastic material as shown in Figure 1. There are two forces H,V acting on the frame. The elements in the set of forces may vary propotionaly with the same scalar factor α. The objective of limit analysis is to find the maximum of α at which the frame is collapses by plastic flow. There are two approaches to find the limit load factor α, the static approach and kinematic approach.

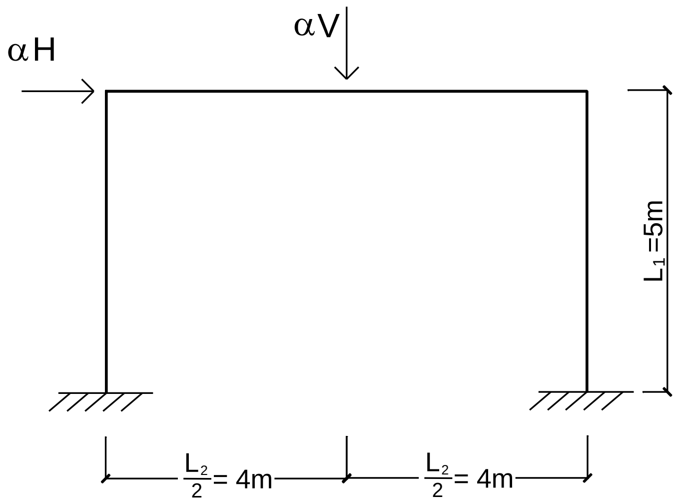

Figure 1: Portal frame subjected two forces

2.1. Static approach

The static approach is based on the lower bound theorem of limit analysis, according to which the safety factor is obtained as the maximum statically admissible load multiplier. This task leads to solving a maximum nonlinear optimization problem

$$\alpha_{\lim} = \max \alpha^{-}$$
$$\text{s.t.:} \begin{cases} -\boldsymbol{\sigma} \cdot \nabla = \alpha^{-} \bar{\mathbf{b}} & \text{in } V \\ \boldsymbol{\sigma} \cdot \mathbf{n} = \alpha^{-} \bar{\mathbf{t}} & \text{on } S_t \\ f(\boldsymbol{\sigma}) \le 0 & \text{in } V \end{cases} \tag{1}$$

The constraints in (1) are the Cauchy equations of equilibrium, static boundary conditions and conditions of plastic admissibility, respectively.

2.2. **Kinematic approach**

The kinematic or upper bound theorem states that the structure fails plastically if the internally (plastically) dissipated power $\dot{W}_{in}$ is less than the power $\dot{W}_{ex}$ of the external loads for all kinematically admissible deformation rates. The limit load factor α^{+} is the smallest kinematic load factor α_{kin} so that the structure fails $\dot{W}_{in} \le \alpha_{kin} \dot{W}_{ex}$ or with normalized $\dot{W}_{ex} = 1$ that $\dot{W}_{in} \le \alpha_{kin}$. The upper bound limit problem can be stated as follows

$$\alpha_{kin} = \min \int_V D(\dot{\boldsymbol{\varepsilon}}) dV$$
$$\text{s.t.:} \begin{cases} \dot{\boldsymbol{\varepsilon}} = \left(\nabla \dot{\mathbf{u}}\right)_{sym} & \text{in } V \\ \dot{\mathbf{u}} = \mathbf{0} & \text{on } S_u \\ \dot{W}_{ex} = \int_V \overline{\mathbf{b}} \cdot \dot{\mathbf{u}}\, \mathrm{dV} + \int_V \overline{\mathbf{t}} \cdot \dot{\mathbf{u}}\, \mathrm{dS} = 1 \end{cases} \tag{2}$$

The finite element method is used to discretize the problem (2). The frame structure of Figure 1 is divided into ne quadrilateral finite elements. If the von-Mises yield condition is used, the internal dissipated power of the frame is calculated as

$$\sum_{e=1}^{ne} \int_A D\left(\dot{\boldsymbol{\varepsilon}}\right) dA = \frac{2}{\sqrt{3}} s_0 \sqrt{\frac{1}{2} \dot{\boldsymbol{\varepsilon}}^T \mathbf{D} \dot{\boldsymbol{\varepsilon}} + \varepsilon_0^2}\, dA \tag{3}$$

By using Gauss integration technique the plastic dissipated power of frame is computed as

$$\sum_{e=1}^{ne} \int_A D\left(\dot{\boldsymbol{\varepsilon}}\right) dA = \sum_{e=1}^{ne} \frac{2}{\sqrt{3}} s_0 \sqrt{\frac{1}{2} \dot{\boldsymbol{\varepsilon}}^T \mathbf{D} \dot{\boldsymbol{\varepsilon}} + \varepsilon_0^2}\, dA = \sum_{i=1}^{NG} \frac{2}{\sqrt{3}} s_0 \sqrt{\frac{1}{2} \dot{\boldsymbol{\varepsilon}}_i^{\,T} \mathbf{D} \dot{\boldsymbol{\varepsilon}}_i + \varepsilon_0^2} \tag{4}$$

in which s_0 is the yield stress, $\dot{\boldsymbol{\varepsilon}} = \left[\dot{\varepsilon}_{11} \;\; \dot{\varepsilon}_{22} \;\; 2\dot{\varepsilon}_{12}\right]^T$ is the strain rate vector

$$\mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$$

ε_0^2 is a very small positive regularization number,

ne is the number of quadrilateral elements of the FEM model of the frame,

NG is the total number of Gauss points of the frame.

The last constraint in Eq. (2) is the external power, it can be written in the form of stresses and strain rate. Using Gauss integration technique, the external power can be expressed

$$\sum_{e=1}^{ne} \int_A \sigma_{ij} \dot{\varepsilon}_{ij} dA = \sum_{i=1}^{NG} w_i \boldsymbol{\varepsilon}_i^T \boldsymbol{\sigma}_i \tag{5}$$

Finally, the FEM formulation of the problem (2) is as follows

$$\min \sum_{i=1}^{NG} \frac{2}{\sqrt{3}} s_0 \sqrt{\frac{1}{2} \dot{\boldsymbol{\varepsilon}}_i^{\,T} \mathbf{D} \dot{\boldsymbol{\varepsilon}}_i + \varepsilon_0^2}$$
$$s.t. \begin{cases} \dot{\boldsymbol{\varepsilon}}_i = \mathbf{B} \dot{\mathbf{u}} \\ \sum_{i=1}^{NG} w_i \boldsymbol{\varepsilon}_i^T \boldsymbol{\sigma}_i = 1 \end{cases} \tag{6}$$

3 STOCHASTIC MODEL FOR THE LIMIT ANALYSIS PROBLEM OF A FRAME

in this section, we introduce a stochastic limit analysis model based on kinematic approach. The yield stress is an uncertain quantity, which is distributed normally or log-normally. The lognormal distribution assumes only positive strength data.

3.1. Equivalent deterministic formulation

If the yield stress s_0 is a random variable, the objective function of the kinematic problem is a stochastic variable and the limit analysis problem is stochastic programming problem. We can state the problem in such a way that one looks for a minimum lower bound λ of the objective function under the constraint that the probability ψ of violation of that bound is prescribed [2,3]

$$\alpha^+ = \min \lambda$$
$$\text{s.t.}: \begin{cases} \text{Prob}\left(\sqrt{\frac{2}{3}} s_0(\omega) \sqrt{\dot{\boldsymbol{\varepsilon}}_i^T \mathbf{D} \dot{\boldsymbol{\varepsilon}}_i + \varepsilon_0^2} \geq \lambda \right) = \psi \\ \dot{\boldsymbol{\varepsilon}}_i = \mathbf{B}\dot{\mathbf{u}} \\ \sum_{i=1}^{NG} w_i \boldsymbol{\varepsilon}_i^T \boldsymbol{\sigma}_i = 1 \end{cases} \tag{7}$$

For sake of simplicity of notation, we denote the plastic dissipation

$$X(\omega) = \sqrt{\frac{2}{3}} s_0(\omega) \sqrt{\dot{\boldsymbol{\varepsilon}}_i^T \mathbf{D} \dot{\boldsymbol{\varepsilon}}_i + \varepsilon_0^2} \tag{8}$$

Now the first constraint of (7) can be rewritten as:

$$\text{Prob}(X \geq \lambda) = 1 - \text{Prob}(X \leq \lambda) = 1 - \text{Prob}\left(\frac{X - \mu_X}{\sigma_X} \leq \frac{\lambda - \mu_X}{\sigma_X} \right) = \psi \tag{9}$$

In (9), μ_X, σ_X are mean value and standard deviation of $\theta(\omega)$. We can see in the inequality

$$\frac{X - \mu_X}{\sigma_X} \leq \frac{\lambda - \mu_X}{\sigma_X} \tag{10}$$

that the left hand side is the normalized random variable with zero mean and unit variance. Using the property $\Phi(-x) = 1 - \Phi(x)$ of the cumulative distribution function (c.d.f.) of the standard normal distribution, the probabilistic condition (9) is replaced by

$$\psi = 1 - \Phi\left(\frac{\lambda - \mu_X}{\sigma_X} \right) = \Phi\left(\frac{\mu_X - \lambda}{\sigma_X} \right) \tag{11}$$

Setting $\psi = \Phi(\kappa)$ we have $\Phi^{-1}(\psi) = \kappa = \dfrac{\mu_X - \lambda}{\sigma_X}$ or $\mu_X - \kappa\sigma_X = \lambda$. Therefore, the chance-constrained program with individual constraints (7) has the deterministic equivalent with $\kappa = \Phi^{-1}(\psi)$:

$$\alpha^{+} = \min \lambda$$
$$\text{s.t.:} \begin{cases} \lambda = \mu_X - \kappa \sigma_X \\ \dot{\boldsymbol{\varepsilon}}_i = \mathbf{B}\dot{\mathbf{u}} \\ \sum_{i=1}^{NG} w_i \boldsymbol{\varepsilon}_i^T \boldsymbol{\sigma}_i = 1 \end{cases} \tag{12}$$

The mean value of $X(\omega)$ is as follows:

$$\mu_{X(\omega)} = \sqrt{\frac{2}{3}} \overline{s}_i \sqrt{\dot{\boldsymbol{\varepsilon}}_i^{\,T} \mathbf{D} \dot{\boldsymbol{\varepsilon}}_i + \varepsilon_0^2} \tag{13}$$

where $\overline{s}_i$ is the mean value of the yield stress $s_{0i}(\omega)$. The variance and standard deviation of $X(\omega)$ are computed as

$$\begin{aligned} \operatorname{Var}\left[X(\omega)\right] &= \operatorname{Var}\left[\sum_{i=1}^{NG} \sqrt{\frac{2}{3}} s_{0i}(\omega) \sqrt{\dot{\boldsymbol{\varepsilon}}_i^{\,T} \mathbf{D} \dot{\boldsymbol{\varepsilon}}_i + \varepsilon_0^2}\right] = \\ &= \left(\sqrt{\dot{\boldsymbol{\varepsilon}}_i^{\,T} \mathbf{D} \dot{\boldsymbol{\varepsilon}}_i + \varepsilon_0^2}\right)^2 \operatorname{Var}\left[\sum_{i=1}^{NG} \sqrt{\frac{2}{3}} s_{0i}(\omega)\right] \end{aligned} \tag{14}$$

Thus we have:

$$\sigma_{X(\omega)} = \sqrt{\operatorname{Var}\left[X(\omega)\right]} = \sum_{i=1}^{NG} \sqrt{\frac{2}{3}} \sigma\left(s_{0i}\right) \sqrt{\dot{\boldsymbol{\varepsilon}}_i^{\,T} \mathbf{D} \dot{\boldsymbol{\varepsilon}}_i + \varepsilon_0^2} \tag{15}$$

In (15), $\sigma\left(s_{0i}\right)$ is the standard deviation of the yield stress and we can denote it as d_i.

Finally, we can write the discretized upper bound of limit load moving the chance constraint to the objective function for the case of normally distributed strength $s_0 \sim \mathcal{N}(\overline{s}_i, d_i^2)$:

$$\alpha^{+} = \min \sum_{i=1}^{NG} \sqrt{\frac{2}{3}} \left(\overline{s}_i - \kappa_i d_i\right) \sqrt{\dot{\boldsymbol{\varepsilon}}_i^{\,T} \mathbf{D} \dot{\boldsymbol{\varepsilon}}_i + \varepsilon_0^2}$$
$$\text{s.t.:} \begin{cases} \dot{\boldsymbol{\varepsilon}}_i = \mathbf{B}\dot{\mathbf{u}} \\ \sum_{i=1}^{NG} w_i \boldsymbol{\varepsilon}_i^T \boldsymbol{\sigma}_i = 1 \end{cases} \tag{16}$$

If the yield stress is lognormally distributed $\ln s_0 \sim \mathcal{N}(\overline{s}_i, d_i^2)$, the deterministic equivalent can be obtained after some transformations:

$$\alpha^{+} = \min \sum_{i=1}^{NG} \sqrt{\frac{2}{3}} e^{(\mu_i - \kappa\sigma_i)} \sqrt{\dot{\boldsymbol{\varepsilon}}_i^{\,T} \mathbf{D} \dot{\boldsymbol{\varepsilon}}_i + \varepsilon_0^2}$$

$$\text{s.t.}: \begin{cases} \dot{\boldsymbol{\varepsilon}}_i = \mathbf{B}\dot{\mathbf{u}} \\ \sum_{i=1}^{NG} w_i \boldsymbol{\varepsilon}_i^T \boldsymbol{\sigma}_i = 1 \end{cases} \tag{17}$$

In (17) μ_i, σ_i are the parameters

$$\mu = \ln\left(E[s_0] \Big/ \sqrt{\frac{Var(s_0)}{E^2[s_0]} + 1} \right), \qquad \sigma = \sqrt{\ln\left(\frac{Var(s_0)}{E^2[s_0]} + 1 \right)} \tag{18}$$

of lognormally distributed yield stress.

3.2 Algorithm to solve the upper bound limit problem of frame structures

For convenient computation, some new varialbles are introduced:

$\dot{\mathbf{e}}_i = w_i \mathbf{D}^{1/2} \dot{\boldsymbol{\varepsilon}}_i$, $\mathbf{t}_i = \mathbf{D}^{-1/2} \boldsymbol{\sigma}_i^E$, $\hat{\mathbf{B}} = w_i \mathbf{D}^{1/2} \mathbf{B}_i$,

$\mathbf{D}^{1/2}$ and $\mathbf{D}^{-1/2}$ are symmetric matrixes which satisfy $\left(\mathbf{D}^{1/2}\right)^{-1} = \mathbf{D}^{-1/2}$ and $\mathbf{D} = \left(\mathbf{D}^{1/2}\mathbf{D}^{1/2}\right)$

.Substituting these variables into (16) we have

$$\alpha^{+} = \min \sum_{i=1}^{NG} \sqrt{\frac{2}{3}} \left(\bar{s}_i - \kappa d_i\right) \sqrt{\mathbf{e}_i^T \mathbf{e}_i + \varepsilon_0^2}$$

$$\text{s.t.} \begin{cases} \dot{\mathbf{e}}_i = \hat{\mathbf{B}}\dot{\mathbf{u}} \\ \sum_{i=1}^{NG} \dot{\mathbf{e}}_i^T \mathbf{t}_i = 1 \end{cases} \tag{19}$$

The penalty method and the Lagrange method are used simultaneously to convert the problem (19) into an unconstrained programming. The penalty function is written as

$$F_p = \sum_{i=1}^{NG} \left(\bar{s}_i - \kappa d_i\right) \sqrt{\dot{\mathbf{e}}_i^T \dot{\mathbf{e}}_i + \varepsilon_0^2} + \frac{c}{2} \left(\dot{\mathbf{e}}_i - \hat{\mathbf{B}}_i \dot{\mathbf{u}}\right)^T \left(\dot{\mathbf{e}}_i - \hat{\mathbf{B}}_i \dot{\mathbf{u}}\right) \tag{20}$$

here c is a penalty parameter, $c \gg 1$.

Problem (19) now becomes

$$\alpha^{+} = \min F_p$$

$$\text{s.t.:} \sum_{i=1}^{NG} \dot{\mathbf{e}}_i^T \mathbf{t}_i = 1 \tag{21}$$

By using the Lagrange multiplier method the problem (21) is converted into an unconstrained programming problem with the Lagarange function:

$$L = F_p - \alpha\left[\sum_{i=1}^{NG} \dot{\mathbf{e}}_i^T \mathbf{t}_i - 1\right] \tag{22}$$

Here α is the Lagrange multiplier.

The *Karush–Kuhn–Tucker conditions (KKT optimality conditions)* gives the equations to obtain optimal solution

$$\begin{cases} \dfrac{\partial L}{\partial \dot{\mathbf{e}}_i} = \left(s_{0i} \dfrac{\dot{\mathbf{e}}_i}{\sqrt{\dot{\mathbf{e}}_i^T \dot{\mathbf{e}}_i + \varepsilon^2}} \right) + c\left(\dot{\mathbf{e}}_i - \mathbf{B}_i \dot{\mathbf{u}}\right) + \alpha \mathbf{t}_i = 0 & (a) \\ \dfrac{\partial L}{\partial \dot{\mathbf{u}}} = -c\mathbf{B}_i^T \left(\dot{\mathbf{e}}_i - \mathbf{B}_i \dot{\mathbf{u}}\right) = 0 & (b) \\ \dfrac{\partial L}{\partial \alpha} = \sum_{i=1}^{NG} \dot{\mathbf{e}}_i^T \mathbf{t}_i - 1 = 0 & (c) \end{cases} \tag{23}$$

The Newton method is applied to solve the system (23) of nonliner equations. We get:

$$\begin{aligned} d\dot{\mathbf{u}} &= d\dot{\mathbf{u}}_1 + \left(\alpha + d\alpha\right) d\dot{\mathbf{u}}_2 \\ d\dot{\mathbf{e}}_i &= \left(d\dot{\mathbf{e}}_i\right)_1 + \left(\alpha + d\alpha\right)\left(d\dot{\mathbf{e}}_i\right)_2 \\ \alpha + d\alpha &= \left(1 - \sum_{k=1}^{NG}\left(\dot{\mathbf{e}}_i^T \mathbf{t}_\mathbf{i}\right) - \sum_{i=1}^{NG}\left(\mathbf{t}_i^T\right)\left(d\mathbf{e}_i\right)_1\right) \Big/ \sum_{i=1}^{NG}\left(\mathbf{t}_i^T\right)\left(d\mathbf{e}_i\right)_2 \end{aligned} \tag{24}$$

in which

$$d\dot{\mathbf{u}}_1 = -\dot{\mathbf{u}} \qquad d\dot{\mathbf{u}}_2 = \mathbf{S}^{-1}\mathbf{f}_{u2} \qquad (d\dot{\mathbf{e}}_i)_1 = -\dot{\mathbf{e}}_i$$

$$(d\dot{\mathbf{e}}_i)_2 = \mathbf{H}_i^{-1}\sqrt{\dot{\mathbf{e}}_i^T \dot{\mathbf{e}}_i + \varepsilon_0^2}\,\mathbf{t}_i + \sqrt{\dot{\mathbf{e}}_i^T \dot{\mathbf{e}}_i + \varepsilon_0^2}\,\mathbf{H}_i^{-1}\mathbf{E}_i^{-1}\hat{\mathbf{B}}_i d\dot{\mathbf{u}}_2 - \sqrt{\dot{\mathbf{e}}_i^T \dot{\mathbf{e}}_i + \varepsilon_0^2}\,\mathbf{H}_i^{-1}\mathbf{E}_i^{-1}\mathbf{H}_i^{-1}\sqrt{\dot{\mathbf{e}}_i^T \dot{\mathbf{e}}_i + \varepsilon_0^2}\,\mathbf{t}_i$$

$$\mathbf{H}_i \approx \left(\sigma_{Yi}\sqrt{\dot{\mathbf{e}}_i^T \dot{\mathbf{e}}_i + \varepsilon^2}\right)\mathbf{I}_i \qquad \mathbf{S} = \sum_{i=1}^{NG} \hat{\mathbf{B}}_i^T \mathbf{E}_i^{-1} \hat{\mathbf{B}}_i$$

$$\mathbf{f}_{u2} = \sum_{i=1}^{NG} \hat{\mathbf{B}}_i \mathbf{E}_i^{-1} \mathbf{H}_i^{-1} \sqrt{\dot{\mathbf{e}}_i^T \dot{\mathbf{e}}_i + \varepsilon_0^2}\,\mathbf{t}_i \qquad \mathbf{E}_i = \left(\frac{\mathbf{I}_i}{c} + \sqrt{\dot{\mathbf{e}}_i^T \dot{\mathbf{e}}_i + \varepsilon_0^2}\,\mathbf{H}_i^{-1}\right)$$

The detailed Algorithm can be found in [2,3].

4 NUMERICAL EXAMPLES

Consider a portal frame clamped at the bases as shown in Figure 1. The reliability analysis is performed in [5] for lengths $l_1 = h = 5$ m, $l_2 = 2l = 8$ m, loads $H = 4kN$, $V = 8kN$. Mullions and transoms have random plastic moments characterized by mean values $\bar{M}_p = 12$ kN and the standard deviations $\sigma_{Mp} = 1.2$kNm. With (18) the parameters for the lognormally distributed plastic moment are $\mu = 2.4799$ and $\sigma = 0.0998$.

Let us determine the limit load factor α for the reliability level $\psi = 0.9999$ (failure probability $P_f = 1 - \psi = 10^{-4}$) so that $\kappa = \Phi^{-1}(\psi) = \Phi^{-1}(0.9999) = 3.71902$.

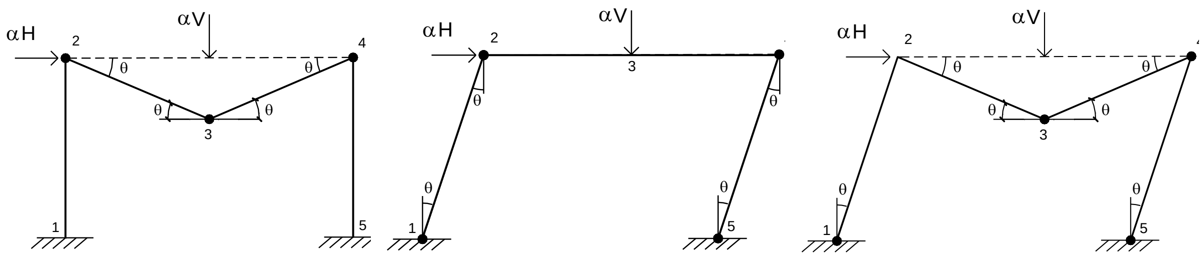

Figure 2: Collapse mechanism of the portal frame: (a) partial beam collapse, (b) sway collapse, (c) combined collapse.

The portal frame has the indeterminacy three. Therefore, a complete collapse mechanism requires four plastic hinges. By the kinematic theorem we can compute the analytical upper bounds from the collapse mechanisms in Figure 2:

Mechanism 1: beam mechanism, partial collapse with three plastic hinges.

$$\alpha_1 V \cdot \dot{v} = \left(M_{p2} + 2M_{p3} + M_{p4}\right)\dot{\theta} = 4M_p\dot{\theta}$$

$$\dot{h} = 0, \qquad \dot{v} = \frac{l_2}{2}\dot{\theta}$$

$$\rightarrow \alpha_1 = \left(4M_p\right) \Big/ \left(V\frac{l_2}{2}\right) = \frac{4 \cdot 12}{8 \cdot 4} = 1.5$$

With chance constraints for normally distributed s_0

$$\alpha_1 = 4\left(\mu_p - \kappa \cdot \sigma_p\right) \Big/ \left(V\frac{l_2}{2}\right) = \frac{4\left(12 - 3.719 \cdot 1.2\right)}{8 \cdot 4} = 0.9422$$

and for lognormally distributed s_0

$$\alpha_1 = 4e^{\left(\mu_p - \kappa \cdot \sigma_p\right)} \Big/ \left(V\frac{l_2}{2}\right) = \frac{4e^{\left(2.4799 - 3.719 \cdot 0.0998\right)}}{8 \cdot 4} = 1.03$$

Mechanism 2: sway mechanism, plastic collapse with four plastic hinges.

$$\alpha_2 H \cdot \dot{h} = \left(M_{p1} + M_{p2} + M_{p4} + M_{p5} \right) \dot{\theta} = 4 M_p \dot{\theta}$$
$$\dot{h} = l_1 \dot{\theta}, \quad \dot{v} = 0$$
$$\rightarrow \alpha_2 = \frac{4 M_p}{H l_1} = \frac{4 \cdot 12}{4 \cdot 5} = 2.4$$

With chance constraints for normally distributed s_0

$$\alpha_2 = \frac{4\left(\mu_p - \kappa \cdot \sigma_p\right)}{H l_1} = \frac{4(12 - 3.719 \cdot 1.2)}{4 \cdot 5} = 1.5074$$

and for lognormally distributed s_0

$$\alpha_2 = \frac{4 e^{\left(\mu_p - \kappa \cdot \sigma_p\right)}}{H l_1} = \frac{4 e^{(2.4799 - 3.719 \cdot 0.0998)}}{4 \cdot 5} = 1.6479$$

Mechanism 3: combined mechanism, plastic collapse with four plastic hinges.

$$\alpha_3 \left(H \cdot \dot{h} + V \cdot \dot{v} \right) = \left(M_{p1} + 2 M_{p3} + 2 M_{p4} + M_{p5} \right) \dot{\theta} = 6 M_p \dot{\theta}$$
$$\dot{h} = l_1 \dot{\theta}, \quad \dot{v} = \frac{l_2}{2} \dot{\theta}$$
$$\rightarrow \alpha_3 = \left(6 M_p \right) \Big/ \left(H l_1 + V \frac{l_2}{2} \right) = \frac{6 \cdot 12}{4 \cdot 5 + 8 \cdot 4} = 1.3846$$

With chance constraints for normally distributed s_0

$$\alpha_3 = 6\left(\mu_p - \kappa \cdot \sigma_p\right) \Big/ \left(H l_1 + V \frac{l_2}{2} \right) = \frac{6(12 - 3.719 \cdot 1.2)}{4 \cdot 5 + 8 \cdot 4} = 0.8697$$

and for lognormally distributed s_0

$$\alpha_3 = 6 e^{\left(\mu_p - \kappa \cdot \sigma_p\right)} \Big/ \left(H l_1 + V \frac{l_2}{2} \right) = \frac{6 e^{(2.4799 - 3.719 \cdot 0.0998)}}{4 \cdot 5 + 8 \cdot 4} = 0.9507$$

The deterministic limit load factor of the frame model is $\alpha = \min\{\alpha_1, \alpha_2, \alpha_3\} = \alpha_3 = 1.3846$ and relates to the combined mechanism. The probabilistic limit load factor is $\alpha = \min\{\alpha_1, \alpha_2, \alpha_3\} = \alpha_3 = 0.8697$ and $\alpha = \min\{\alpha_1, \alpha_2, \alpha_3\} = \alpha_3 = 0.9507$ if the yield stress assumes a normal and a lognormal distribution respectively. It also relates to the combined mechanism.

For the FEM solution, the frame is modelled by 592 quadrilateral element. The size of the cross sections of mullions and transom are the same (1 cm x 8 cm). In the FEM model, $L_2 = 8\text{m}$ is counted between the neutral axes of the mullions as in Figure 3. It is assumed that the yield stress s_0 is the only source of the uncertainty of the plastic moment M_p so that the mean value is $\bar{s}_i = \bar{M}_P \cdot 4/(bh^2) = 12\text{kNm} \cdot 4/(0.01\text{m} \cdot 0.08^2\,\text{m}^2) = 750\text{Nmm}^{-2}$ and the standard deviation is $d_i = 0.1\bar{s}_i = 75\text{Nmm}^{-2}$. Results are shown in Table 1 and Figure 5.

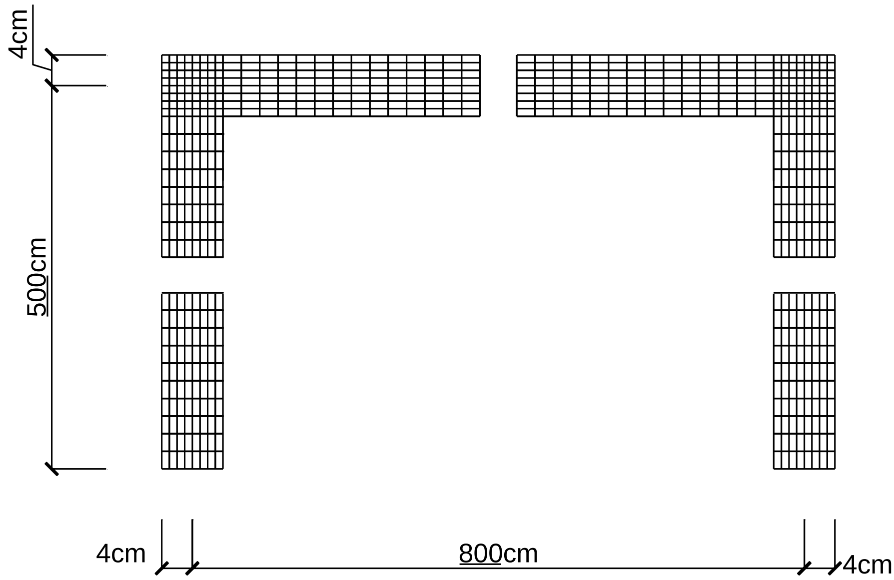

Figure 3: FEM mesh with 592 quadrilateral elements. The gaps in the structure show that he beams are longer and the dimensions that are not to scale are underlined.

Table 1: Limit load factors for two models and different reliability levels

Reliability level ψ	Failure prob. P_f	Frame model			FEM model		
		Determinist.	Normal	Lognormal	Determinist.	Normal	Lognormal
0.9999	10^{-4}	1.3846	0.8697	0.9505	1.52	0.94	1.09
0.999	10^{-3}		0.9567	1.0121		1.08	1.18
0.99	10^{-2}		1.0625	1.0923		1.15	1.26
0.5	0.5		1.3846	1.3778		1.52	-

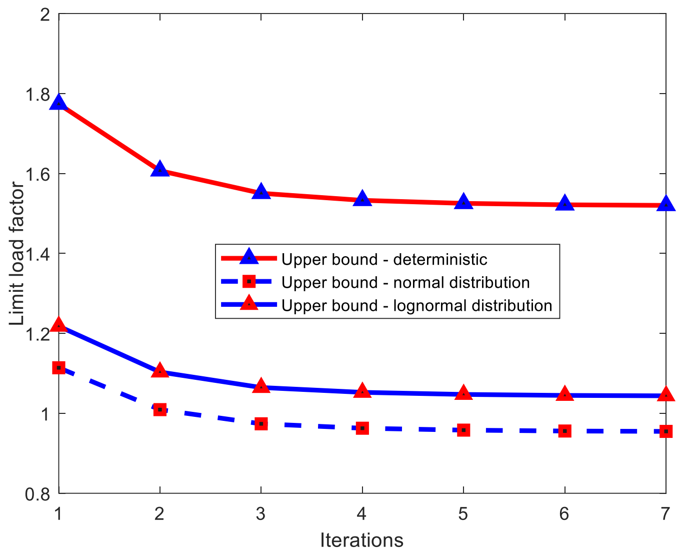

Figure 4: Convergence of limit load factors for reliability level $\psi = 0.9999$ (failure probability $P_f = 10^{-4}$)

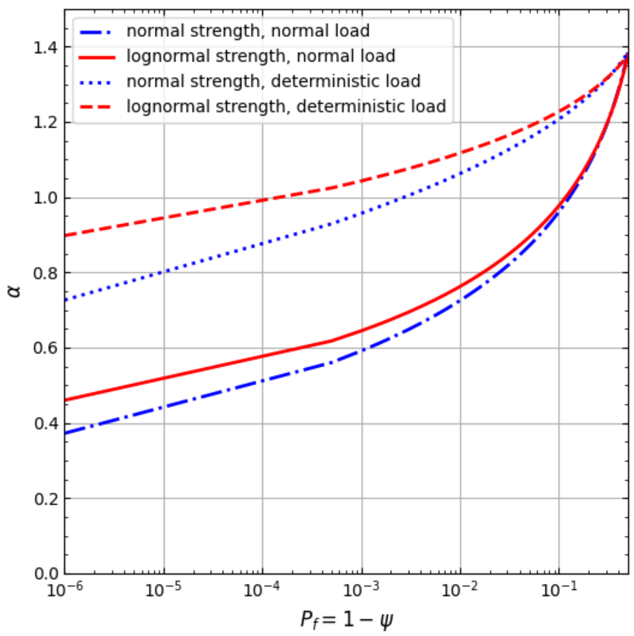

Figure 5: Limit loads for various stochastic models (with random loads are not discussed here) with data of [5].

The probabilistic α assumes the determinstic α for the normal distribution if $\psi = P_f = 0.5$. The deterministic limit load factor of the FEM model is $\alpha = 1.52$, which is larger than the limit load factor $\alpha = 1.385$ of the frame model. The same tendency holds for the pobabilitistic limit load factors for both distributions of the yield stress. The FEM limit analysis converges in 5-6 iterations. Values $\alpha < 1$ indicate that the structure is already overloaded for the chosen reliability level. The failure probabilty $P_{f1} = 1.187 \cdot 10^{-2}$ of the partial beam collapse and $P_{f3} = 0.877 \cdot 10^{-2}$ of the combined collapse is calculated for the frame model with the First Order Reliability Method (FORM) in [5] for the case that also the loading is uncertain. All mechanisms contribute to the total failure probability. The lower bound $\min P_f = \max\left\{P_{f1}, P_{f2}, P_{f3}\right\} = P_{f1} = 1.187 \cdot 10^{-2}$ is assumed if the mechanisms are fully positively correlated. The upper bound $\max P_f = 1 - \left(1 - P_{f1}\right) \cdot \left(1 - P_{f2}\right) \cdot \left(1 - P_{f3}\right) = 2.054 \cdot 10^{-2}$ is assumed if the mechanisms are completely uncorrelated. The order of magnitude is similar to the results in Table 1. Currently our formulation with individual chance constraints cannot combine contributions of the different mechanisms. The lognormal distribution is preferred over the normal distribution for the plastic moment and for yield stress because they are always positive.

5 CONCLUSIONS

Probabilistic limit analysis can be made with the deterministic equivalent of the chance constraints for normally or lognormally distributed strength data. Then the analysis is basically the same as a deterministic limit analysis. The limit loads are obtained for any target reliability level, if the mean value and standard deviation of strength are available. The limit load has to be decreased greatly if a high reliability of the structure is required. Any statically indeterminate frame structure can be handled in the demonstrated way. The extension to uncertain loading can be done as shown in [3]. Future work has to be addressed to the question how a possible relevant contribution of several collapse mechanisms can be included in the analysis.

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