Unified formulation of a family of iterative solvers for power systems analysis

Domenico Borzacchiello, Muhammad H. Malik, Francisco Chinesta, Raquel García-Blanco and Pedro Díez

Abstract—This paper illustrates the construction of a new class of iterative solvers for power flow calculations based on the method of alternating search directions. This method is fit to the particular algebraic structure of the power flow problem resulting from the combination of a globally linear set of equations and nonlinear local relations imposed by power conversion devices, such as loads and generators. The choice of the search directions is shown to be crucial for improving the overall robustness of the solver. A noteworthy advantage is that constant search directions yield stationary methods that, in contrast with Newton or Quasi-Newton methods, do not require the evaluation of the Jacobian matrix. Such directions can be elected to enforce the convergence to the high voltage solution. The method is explained through an intuitive example illustrating how the proposed generalized formulation is able to include other nonlinear solvers that are classically used for power flow analysis, thus offering a unified view on the topic. Numerical experiments on the IEEE 8500-node distribution feeder benchmark are carried out to assess the performances.

Index Terms—Power System Simulation, Power Flow Analysis, Alternating Search Directions, Iterative Solver.

I. INTRODUCTION

THE power flow problem consists in determining the state of a power system in terms of voltage magnitudes and phase angles at each bus, for given load and generation profiles. This can be achieved through the solution of a set of nonlinear power equilibrium equations by means of a numerical iterative method. Considerable research effort has been put into the development of numerical techniques to solve this problem, many of which have come to the point of being considered as "milestones" of power system simulation and are now extensively used by the power industry [1]. Nonetheless, the ever-evolving technological scenario characterizing the power engineering domain demands for a constant improvement of the numerical methods, in order to keep pace with the new standards of robustness, computational speed and reliability required in simulation tools. This idea is what motivates the research work behind this paper.

A. Related work

Historically, power flow studies started with Gauss-Seidel (GS) type methods [2], [3], Newton-Raphson's methods (NR) [4], [5], or fixed point algorithms based on the admittance

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or impedance matrix, like the Implicit Z bus method (IZB) [6], [7]. Despite their flexibility and low memory usage, GS methods have low convergence rates compared to NR methods, who enjoy optimal quadratic convergence but come with an increased computational cost due to the need of assembling and solving the Jacobian system at each iteration. The Implicit Z bus method has a good convergence rate and avoids the problem of reforming a different linear system at each iteration, but it tends to perform less efficiently when several PV nodes are in the system. [8], [9], [10].

A variety of formulations of NR have been developed in order to address the problem of Jacobian update. These include Newton-Krylov methods [11], Jacobian-free [12], or partial Jacobian update variants [13]. Among the most popular approaches is the Fast Decoupled Load Flow Method (FDLF) [14], providing an approximation of the Jacobian based on practical properties of the power flow problem. In this way Newton's method is reduced to a sequence of decoupled linear problems for the voltage magnitude and phase angle, whose matrices are kept constant throughout the iterations. The theoretical background of this method has been elucidated from a mathematical viewpoint in subsequent works [15], [16].

A major drawback of Newton's and Quasi-Newton's methods is the inability to systematically select the operative solution among the multiple possible solutions of the nonlinear set of equation governing the power flow. It is known that convergence behavior of NR is strongly related to the choice of the initial guess solution and that the method may converge to a spurious non operative solution or simply fail to converge in some cases. This situation is especially critical when the system is close to its voltage stability margin. Different alternatives exist to overcome this difficulty, like numerical continuation techniques [17], [18], methods based on truncated Taylor expansions [19], [20], [21] or analytical continuation like the more recent Holomorphic Embedding Load Flow Method (HELM) [22], relying on Padé's approximants.

While these techniques are able to enforce the convergence to the operative solution and are computationally fast, they have less flexible modeling capabilities. For instance there are reported difficulties in modeling PV nodes in IZB [23] or HELM [24], whereas this is straightforward in NR method. One possibility is to include all control actions, including voltage control and limit enforcement, in an additional loop external to the power flow solution. This inevitably leads to more iterations since for each control iteration a power flow has to be solved. On the other hand, this strategy also reflects more closely the way real power systems are operated.

B. Contribution

With the present work we introduce a new class of algorithms that are designed to enforce the convergence to operative high voltage solutions, while retaining a relatively simple structure without the need of evaluating and factorizing the Jacobian matrix. This "family" of solvers is specifically tailored for the algebraic structure of the system of equations arising from the formulation of the power flow problem, and is defined by two free parameters that can be geometrically interpreted as search directions, as will be explained later. Most importantly, the proposed approach gives a unified formulation for a class of power flow iterative methods. Indeed it will be shown how some of the classic methods can be obtained from specific choices of the search directions.

C. Organization of the paper

The layout of the paper is organized as follows: the power flow equations are reviewed in section II. Here the notation is also set to make the present paper self contained. The development of the new solver is illustrated in detail in section III. Examples are presented in section IV where performance issues and treatment of voltage controlled nodes are also discussed. Finally, conclusions are drawn in section V.

II. THE POWER FLOW PROBLEM

An electrical network can be represented as an undirected graph comprising a set of $N_b + 1$ nodes and N_e edges. Each node of the graph represents a bus, specified by the integer $j = 0, 1, 2, ..., N_b$, while every edge is a transmission line corresponding to the pair (j, k). Information on both the grid topology and the characteristics of its power delivery devices is entirely enclosed in the nodal admittance matrix $\boldsymbol{Y}$. This choice of representation can accommodate models of capacitors, transformers, constant impedance loads from ZIP models as well as pi-models for the lines (short, medium and long) with shunt capacitance. Note that bold letters are used to denote matrices, while capital regular letters denote vectors.

Writing Kirchhoff's current law at any node the following algebraic linear system is obtained:

$$\boldsymbol{Y}V = I_0 + I, \tag{1}$$

where V is the vector of unknown voltage phasors evaluated at each node, I_0 is the vector containing the constant current part of the ZIP model and I the vector of currents injected in (positive) or withdrawn (negative) from the nodes due to a constant power source S. The notation $\boldsymbol{Y}V$ stands for the matrix-vector product. Currents, voltages and powers are nonlinearly related through power balance equations, which can be written in vector form as:

$$S = V \odot I^*, \tag{2}$$

with I^* being the complex conjugate of I and the symbol $\odot$ denoting the Hadamard (component-wise) vector product.

In the present development, no particular assumption on the model of lines is made yet, because the algebraic structure possessed by the system (1) is quite general and arises in most

power flow formulations independent of the model adopted. For simplicity, only static load and generator modes are considered in this paper. This does not limit the validity of the proposed method, since the only required hypothesis is that the power at a given bus depend on the current and voltages in a strictly *local* sense. Exponential loads models or combination of polynomials and exponential models also fall in this category.

By incorporating equation (2) into (1), the following non-linear system is obtained:

$$\boldsymbol{Y}V = I_0 + S^* \oslash V^*, \tag{3}$$

with the symbol $\oslash$ denoting the component-wise quotient of vectors. Equation (3) is referred to as the injected current form. Multiplying both right and left hand side by V one obtains the power form:

$$V^* \odot [\boldsymbol{Y}V - I_0] = S^* \tag{4}$$

In this formulation the slack node is transformed into equivalent current sources at adjacent buses, and their contribution is accounted for in the vector I_0, while the corresponding complex equation is eliminated from the system. Therefore, the $\boldsymbol{Y}$ matrix is in general a $n \times n$ complex matrix, while voltages and currents are vectors of $\mathbb{C}^n$, with $n = N_b$ for single-phase systems, or $n = 3N_b$ for three-phase systems.

III. PROPOSED METHODOLOGY

A. The method of alternating search directions

Equation (3) is the combination of linear global problem (1) and nonlinear local constraints (2). In the derivation of the proposed methodology, the first idea is to consider the augmented system formed by equations (1) and (2), instead of the primitive formulation (3), as in [25].

In this framework, a single nonlinear iteration is conceived as the combination of two steps that are obtained by pairing equations (1) and (2) with additional linear relations between voltages and currents, expressing the so-called search directions.

For a given matrix $\boldsymbol{\alpha} \in \mathbb{C}^{n \times n}$ and initial pair $(V, I)^l$, an intermediate solution is found from the linear system

$$\begin{cases} I^{l+\frac{1}{2}} - I^l = \boldsymbol{\alpha}(V^{l+\frac{1}{2}} - V^l) \\ \boldsymbol{Y}V^{l+\frac{1}{2}} = I_0 + I^{l+\frac{1}{2}} \end{cases}. \tag{5}$$

Similarly, in a second step, for a given diagonal matrix $\boldsymbol{\beta} \in \mathbb{C}^{n \times n}$, the solution is updated by solving the system

$$\begin{cases} I^{l+1} - I^{l+\frac{1}{2}} = \boldsymbol{\beta}(V^{l+1} - V^{l+\frac{1}{2}}) \\ V^{l+1^*} \odot I^{l+1} = S^* \end{cases}. \tag{6}$$

We refer to the class of algorithms defined by (6) and (5) as the *Method of Alternating Search Directions*. More detailed theoretical background is provided in [26] where the same method is applied to nonlinear structural mechanics problems.

There are several advantages associated with this approach:

- The non-linearity and the non-locality can be tackled separately in a *divide et impera* fashion, since of the two arising subproblems, (5) is global but linear, while (6) is
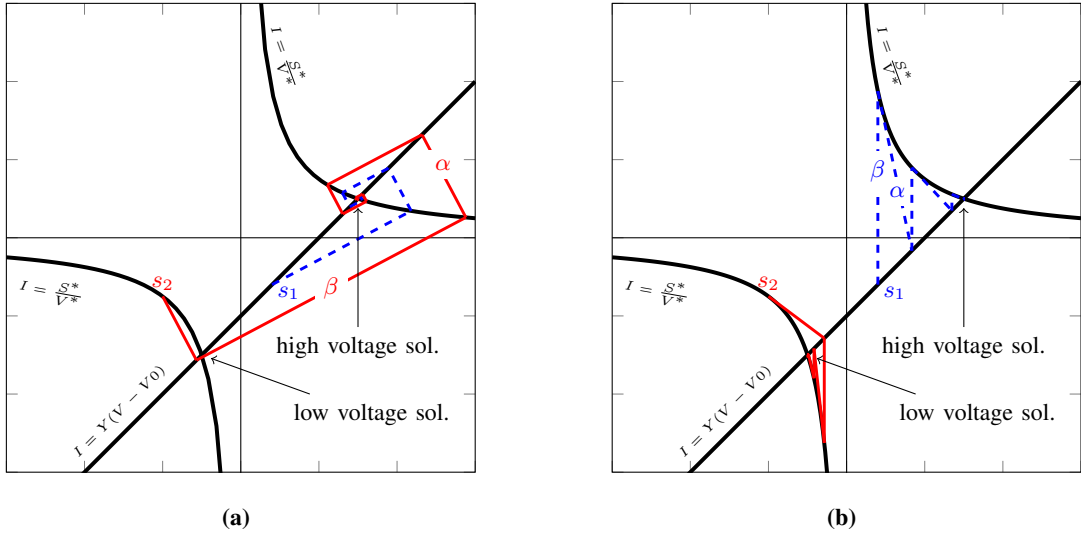
Fig. 1 Graphic comparison between different search directions (α, β). The two trajectories (continuous and dashed lines) are generated by choosing the first solution in the lower branch of the hyperbola (s_2) or on the line close to the upper branch of the hyperbola (s_1). Generic alternating orthogonal directions (a). Newton's method (b).

nonlinear but local, meaning that each nodal equation can be solved individually.

- If α is constant, the matrix factorization needed to solve system (5) is only performed once.
- Equations (6) can be solved analytically (see appendix A). The fact that each equation has two possible roots, reflects the existence of low and high voltage solutions in the power grid. The high voltage solution is generally regarded as a desired operative solution and can be therefore enforced exactly through the whole iterative process by selecting the appropriate root. Hence, the converged solution is by construction an operative solution.
- Since the pairs $(V, I)^{l+\frac{1}{2}}$ and $(V, I)^{l+1}$ fulfill equations (1) and (2) respectively and the algorithm is numerically consistent, then $(V, I)^{l+\frac{1}{2}} \equiv (V, I)^{l+1}$ implies that both $V^{l+\frac{1}{2}}$ and V^{l+1} fulfill equation (3), that is, the solution has converged. This means that the distance between the two substep solutions measured in an appropriate norm $||V^{l+\frac{1}{2}} - V^{l+1}||$ can be naturally used as an error indicator to check convergence.

B. Geometrical interpretation

In order to gain some insight about the logic of the proposed scheme, we apply it to the prototype scalar equation:

$$V = V_0 + Z \frac{S^*}{V^*} . \qquad (7)$$

This can be seen as the power flow equation for a two-bus system with line impedance $Z = R + iX$, slack node voltage V_0 and a load $S = P + iQ$. The analytical solution can be expressed in close form as:

$$\frac{V}{V_0} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + \sigma_R - \sigma_I^2} - i\sigma_I , \qquad (8)$$

where $\sigma_R = (XQ + RP)/V_0^2$ and $\sigma_I = (XP - RQ)/V_0^2$, provided that the determinant $\Delta = 1/4 + \sigma_R - \sigma_I^2$ is non

negative. The operative solution is the high voltage one (the one resulting from the sign plus). For the sake of a clear visual representation let us focus on the case of equation (7) having only real coefficients. In this particular case, the set of solutions of the global linear equilibrium lie on the line of equation $I = Y(V - V_0)$, with $Y = Z^{-1}$, while the nonlinear part is represented by the hyperbola of equation $I = S^*/V^*$. Both curves are shown in figure 1. The high voltage solution is the intercept of the line with the upper branch of the hyperbola. In this case the proposed method can be identified with the following geometrical construction, illustrated in figure 1a: starting from any point on the hyperbola (local problem), we move from there to the line (global problem) following a straight paths (continuous line) with constant slope α and then from the line back to the hyperbola with a straight path of slope β, until the intersection point is found. The same result is obtained for any other initial point either on the hyperbola or on the line. With this perspective, it is possible to see Newton's algorithms as a particularization of the discussed method for α equals to the local tangent to the hyperbola and $\beta \to \infty$. Although Newton's method enjoys a faster convergence rate due to the variable α, it can be noticed from figure 1b that it is impossible to enforce the correct branch of the hyperbola, therefore the final solution depends on the initial guess.

C. Choice of the search directions

When dealing with more complex systems than the one presented in the previous section, currents can be eliminated from the two equations (5) and (6), and the iterative algorithm for V can be formulated as follows:

- **Global Step.** Starting from the iterate V^l, the intermediate solution $V^{l+\frac{1}{2}}$ is found by solving the linear system

$$\left[(\boldsymbol{Y} - \boldsymbol{\alpha}) V^{l+\frac{1}{2}} \right] = S^* \oslash V^{l^*} - \boldsymbol{\alpha} V^l + I_0 . \qquad (9)$$

- **Local Step.** The new iteration V^{l+1} is then obtained from the solution of the system

$$\beta V^{l+1} + \left[(\boldsymbol{Y} - \boldsymbol{\beta}) V^{l+\frac{1}{2}} - I_0 \right] - S^* \oslash V^{l+1^*} = 0 . \quad (10)$$

It is now possible to see some of the classical algorithms in the light of the proposed scheme.

1) Gauss-Seidel: By taking $\boldsymbol{\beta} \to \infty$ and $\boldsymbol{\alpha} = \boldsymbol{Y}_U$, the upper triangular part of $\boldsymbol{Y}$, it follows that $\boldsymbol{Y} - \boldsymbol{\alpha} = \boldsymbol{Y}_L$ is the lower triangular part of $\boldsymbol{Y}$ plus its diagonal and $V^{l+1} \to V^{l+\frac{1}{2}}$. Therefore the scheme is reduced to:

$$V^{l+1} = V^{l+\frac{1}{2}} = \boldsymbol{Y}_L^{-1} \left[S^* \oslash V^{l^*} - \boldsymbol{Y}_U V^l + I_0 \right] . \quad (11)$$

2) Newton-Raphson: The formal equivalence with NR is only possible if the current injection form (3) is used with the rectangular representation of the voltage. This is necessary due to the fact that the injected current is not a holomorphic function of the voltage and complex derivation is not formally defined. By introducing the real and imaginary parts of voltages and currents as separate variables, and recalling that $\boldsymbol{Y} = \boldsymbol{G} + i\boldsymbol{B}$, the following quantities are defined:

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{G} & -\boldsymbol{B} \\ \boldsymbol{B} & \boldsymbol{G} \end{bmatrix} \quad , \quad W = \begin{bmatrix} V_r \\ V_i \end{bmatrix} \quad (12)$$

and

$$N = \begin{bmatrix} N_r \\ N_i \end{bmatrix} = \begin{bmatrix} I_{0_r} + (P \odot V_r + Q \odot V_i) \oslash \left(V_r^2 + V_i^2 \right) \\ I_{0_i} + (P \odot V_i - Q \odot V_r) \oslash \left(V_r^2 + V_i^2 \right) \end{bmatrix} \quad (13)$$

the power flow equations can be written as :

$$MW = N . \quad (14)$$

Taking $\boldsymbol{\beta} \to \infty$ and

$$\boldsymbol{\alpha} = \begin{bmatrix} \frac{\partial N_r}{\partial V_r} & \frac{\partial N_r}{\partial V_i} \\ \frac{\partial N_i}{\partial V_r} & \frac{\partial N_i}{\partial V_i} \end{bmatrix} , \quad (15)$$

it follows that $W^{l+1} \to W^{l+\frac{1}{2}}$ and the resulting algorithm reads as:

$$W^{l+1} = W^{l+\frac{1}{2}} = W^l - (M - \alpha)^{-1} \left(MW^l - N^l \right) . \quad (16)$$

The previous iterative scheme, coincides with the NR formulation found in [27], [5], which is, according to the authors in [5], substantially faster than the power mismatches formulation written in polar coordinates. By keeping the Jacobian matrix $(M - \alpha)$ fixed as the one evaluated at the first iteration it is possible to obtain fast Quasi-Newton algorithms.

3) Implicit Z bus method: This is obtained for $\boldsymbol{\beta} \to \infty$ and $\boldsymbol{\alpha} = 0$. Then the iterative scheme becomes:

$$V^{l+1} = \boldsymbol{Y}^{-1} \left(S^* \oslash V^{l^*} + I_0 \right) . \quad (17)$$

The implication of this choice is that the number of required operations per iteration is significantly reduced, because the local step only requires evaluating the injection currents for each power conversion device using the voltage from the previous iteration. It is interesting to notice that when applied to equation (7), modeling the two-bus system, the iterative scheme reads as:

$$V^{l+1} = V_0 + Z \frac{S^*}{V^{l^*}} . \quad (18)$$

In this particular form, we obtain a fixed point algorithm that can be seen as a continued fraction approximation of the solution, since for $U = V/V_0$ the solution of (7) is given by

$$U = 1 + \cfrac{\sigma}{1 + \cfrac{\sigma^*}{1 + \cfrac{\sigma}{1 + ...}}} , \quad (19)$$

where $\sigma = \frac{ZS^*}{|V_0|^2}$, which is exactly the same continued fraction resulting from the application of the Holomorphic Embedding method to the same system, as demonstrated by constructive proof in [22]. The convergents of this continued fraction corresponds to the application of the fixed point (18). Therefore, the iterative solutions found with the IZB, coincide with the ones found with the HELM as the number of coefficients of the Padé approximant is increased. Deriving a proof of the formal equivalence of the two approaches in the general case involving matrices and vectors is beyond the scope of this work, however it seems useful to report that in all numerical experiments that have been performed on systems with only PQ nodes, both HELM and IZB have been observed to generate a similar sequence of solutions, the difference being most likely due to recursive applications of discrete deconvolutions that are needed in HELM, which are known to give rise to a numerical precision problem [24].

4) Proposed method: In the above paragraphs, the search direction have been selected such as to obtain other existing methods. An alternative choice for $\boldsymbol{\alpha}$ is

$$\boldsymbol{\alpha} = diag \left(S^* \oslash |V_b|^2 \right) , \quad (20)$$

in which case the matrix $(\boldsymbol{Y} - \boldsymbol{\alpha})$ becomes a modified admittance matrix whose diagonal includes the linear part of the loads. The linearization point is chosen as the point of 100% of the base voltage V_b. This choice is optimal in many cases, since well designed grids are normally operating not far from this point. This strategy is also adopted in the open source code Open DSS [28], [29], in which the linear part of the loads is taken into account by an additional diagonal term in the nodal admittance matrix by virtue of the Norton theorem, while only the extra injection current due to the non linear part of the loads is added to the right hand side of the power flow equations. Indeed, the basic solution algorithm of Open DSS can be seen as a particularization of the method of alternating search directions when $\boldsymbol{\alpha} = diag \left(S^* \oslash |V_b|^2 \right)$ and $\boldsymbol{\beta} \to \infty$.

All the methods discussed so far have in common the choice of $\boldsymbol{\beta} \to \infty$, which means in practice that the voltage of the local step is simply inherited from the global step, while the current is calculated using (2). A practical choice for $\boldsymbol{\beta}$ can be based on the following rationale. Given that $V^{l+\frac{1}{2}}$ verifies equation (1), then if $\boldsymbol{\beta} = \boldsymbol{Y} - \boldsymbol{\alpha}$, solving system (6) yields the solution to the power flow problem, that is, the method converges in a single iteration. However, with this choice the decoupling between the nonlinear equations would be lost and solving (6) would be as difficult as solving the original primitive formulation. Reasonably, $\boldsymbol{\beta}$ should be a good spectral approximation of $\boldsymbol{Y}$ while retaining a diagonal structure. Therefore, an option is to take

$$\boldsymbol{\beta} = diag \left(\boldsymbol{Y} - \boldsymbol{\alpha} \right) . \quad (21)$$

In some numerical studies we found that

$$\boldsymbol{\beta} = diag\left(\left(\boldsymbol{Y} - \boldsymbol{\alpha}\right)^{-1}\right)^{-1} \qquad (22)$$

is also a viable choice.

The algorithm can be resumed as follows:

- Assemble the system by forming the matrix $\boldsymbol{Y}$ and the vectors S and I_0
- Select the search directions $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$
- Factorize the matrix $(\boldsymbol{Y} - \boldsymbol{\alpha})$
- Evaluate the initial guess as $V^0 = (\boldsymbol{Y} - \boldsymbol{\alpha})^{-1} I_0$
- Alternate steps (10) and (9) until $||V^{l+\frac{1}{2}} - V^{l+1}||_\infty$ is less than an arbitrary small constant ϵ

Every iteration consists of a backward and a forward substitution for the linear global stage, and the computation of the high voltage root of n decoupled quadratic equations, which can be done exactly (i.e. to the machine precision) without the need of an iterative solver. The initial guess V^0 corresponds to the solution of the systems with no loads and only due to the slack node voltage. Note that this choice is not mandatory and the method can be started in other ways. Convergence is obtained even with arbitrary random starts for which NR diverges. The proposed initial guess allows to set all phase angles in the proper relationships and is relatively inexpensive since the factorization of the matrix $(\boldsymbol{Y} - \boldsymbol{\alpha})$ is readily available.

As a final remark, it should be noted that the matrix $\boldsymbol{Y} - \boldsymbol{\alpha}$ should be non-singular and the search direction should never be parallel, that is, the matrices $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ should not be equal, otherwise the method stagnates at the first iteration. Based on this observation, another possible choice is $\boldsymbol{\beta} = diag\left(\boldsymbol{Y}\right)$ and $\boldsymbol{\alpha} = -\boldsymbol{\beta}^{-1}$ which results in orthogonal search directions.

IV. NUMERICAL RESULTS

In this section numerical examples with transmission and distribution grids are considered. The treatment of PV nodes and reactive power limits is detailed and performance comparisons with other existing methods are shown. Finally, a way of implementing control actions is suggested and the effect of different search direction is discussed for a realistic system modeling a radial distribution feeder.

A. Treatment of PV nodes

The power grids considered for comparison are the IEEE 14 test case, the IEEE 30 test case, the IEEE 57 test case and the 89-bus European high voltage Transmission Network (ETN) [30]. The system data are available from the open source code MatPower [31], which is also used to run the power flow solution through different algorithms. As already mentioned in section I-A, in the framework of methods that use a fixed matrix through the iterations and complex representation of the voltage, such as IZB, it is difficult to impose the voltage levels in PV nodes. The most common practice is to transform PV nodes into PQ nodes and adjust the reactive power levels withing the admissible ranges until the voltage levels are set to the desired values. This strategy implies the presence of an outer iterative loop in which the levels of reactive power are varied and an inner iterative loop to solve the power flow each time the reactive powers are changed.

In contrast with this, the peculiar two-step structure in the method of alternating search directions allows to adjust the reactive power levels inside the power flow loop between the local and the global stages, thus avoiding the necessity of repeating the power flow solution multiple times. In this work we adopt the PV sensitivity matrix method to correct the reactive power levels as proposed [32], with the only difference that this procedure is now embedded in the power flow solver.

The unknown voltages are partitioned into V_1, the voltages corresponding to the PQ nodes, and V_2, the voltages corresponding to the PV nodes. Following the same partitioning, the admittance matrix can be written as:

$$\boldsymbol{Y} = \begin{bmatrix} \boldsymbol{Y}_{11} & \boldsymbol{Y}_{12} \\ \boldsymbol{Y}_{21} & \boldsymbol{Y}_{22} \end{bmatrix}. \qquad (23)$$

At the end of the global step, if the voltage magnitudes $|V_2^{l+\frac{1}{2}}|$ deviate from the desired values E_{PV}, it is possible to calculate the extra injected current in order to compensate the voltages by applying Kron reduction to the block $\boldsymbol{Y}_{22}$:

$$\Delta I_2^{l+\frac{1}{2}} = \left(\boldsymbol{Y}_{22} + \boldsymbol{Y}_{21}\boldsymbol{Y}_{11}^{-1}\boldsymbol{Y}_{12}\right) \Delta V_2^{l+\frac{1}{2}}, \qquad (24)$$

in which $\Delta V_2^{l+\frac{1}{2}}$ is the required voltage increment. In practice this is taken as

$$\Delta V_2^{l+\frac{1}{2}} = \left(E_{PV} - |V_2^{l+\frac{1}{2}}|\right) \odot V_2^{l+\frac{1}{2}} \oslash |V_2^{l+\frac{1}{2}}|, \qquad (25)$$

which implies that only the voltage magnitude is adjusted while the phase angles are left unaltered. The increment in reactive power is computed as $\Delta Q_2^{l+\frac{1}{2}} = \Im(V_2^l \odot \Delta I_2^{l+\frac{1}{2}*})$ and the reactive power levels are corrected as follows:

$$Q_2^{l+1} = Q_2^l + \gamma \Delta Q_2^{l+\frac{1}{2}}, \qquad (26)$$

where γ is an under-relaxation parameter that can be used to accelerate the convergence but also to enforce the respect of the physical reactive limits of the generators. Once the reactive powers are updated for the PV nodes, the voltages V_2^{l+1} are updated using (10) as if they were PQ nodes. Table I, shows the convergence of the solution for the IEEE 14 test case. The first 3 significant digits converge within 5 iterations ($\epsilon = 10^{-3}$), while 10 iterations are needed for the solution to have an accuracy of 5 significant digits ($\epsilon = 10^{-5}$). Search directions were chosen according to expressions (20) and (22).

The number of iterations required to have a 5-digit precision is compared for different iterative methods in table II. The methods NR, GS and FDLF (in both XB and BX versions) were run via MatPower, while IZB was simply adapted from the method of alternating search directions by choosing the search directions as explained in section III-C3. For the considered test cases in precence of PV nodes, the performances of the proposed strategy are comparable to those of FDLF and superior to those of GS and IZB. For this latter method the PV sensitivity matrix method could not be included inside the power flow solution and had to implemented in an outer loop.

Bus	NR		ASDM $\epsilon = 10^{-3}$		ASDM $\epsilon = 10^{-5}$	
	$|V|$ p.u.	δ [°]	$|V|$ p.u.	δ [°]	$|V|$ p.u.	δ [°]
1	1.0600	0	1.0600	0	1.0600	0
2	1.0450	-4.9826	1.0451	-4.9800	1.0450	-4.9826
3	1.0100	-12.725	1.0102	-12.715	1.0100	-12.725
4	1.0177	-10.313	1.0179	-10.305	1.0177	-10.313
5	1.0195	-8.7739	1.0197	-8.7677	1.0195	-8.7739
6	1.0700	-14.221	1.0704	-14.207	1.0700	-14.221
7	1.0615	-13.360	1.0618	-13.347	1.0615	-13.360
8	1.0900	-13.360	1.0903	-13.347	1.0900	-13.360
9	1.0559	-14.939	1.0563	-14.923	1.0559	-14.939
10	1.0510	-15.097	1.0513	-15.082	1.0510	-15.097
11	1.0569	-14.791	1.0573	-14.775	1.0569	-14.791
12	1.0552	-15.076	1.0556	-15.060	1.0552	-15.076
13	1.0504	-15.156	1.0508	-15.140	1.0504	-15.156
14	1.0355	-16.034	1.0359	-16.016	1.0355	-16.034

TABLE I   Convergence of the solution obtained with the method of Alternating Search Directions (ASDM). The reference solution is obtained with Newton-Raphson's method using MatPower.

Grid	NR	FDXB	FDBX	GS	IZB	ASDM
IEEE 14	3	8+7	6+5	181	18 (20)	9 (14)
IEEE 30	3	8+7	6+5	473	25 (26)	12 (13)
IEEE 57	4	7+6	7+6	588	14 (25)	9 (15)
ETN	4	8+7	8+7	-	34 (43)	14 (13)

TABLE II   Comparative table showing the number of iterations required to have a 5-digit accurate solution for different benchmarks and numerical methods: Newton-Raphson (NR), Fast Decoupled Load FLow (FDLF) in both XB and BX versions, Gauss-Seidel (GS), Implicit Z bus method (IZB) and method of Alternating Search Directions (ASDM). The iterations listed for FDLF include Q- and P-iterations, IZB and ASDM include the iterations for the flat start (in parenthesis).

B. IEEE 8500-Node Feeder Benchmark

From the results of the previous section, NR stands out as the method with the fastest convergence. However, when applied to large systems the solution of the Jacobian system at every iteration may become very expensive in terms of CPU time, since the cost of solving a linear systems of equations scales with n^3. To illustrate this case, we consider the IEEE 8500-node benchmark [33].

This radial distribution feeder consists of approximately 4800 buses that are single-, two or three- phase. The total number of nodes, and therefore voltage unknowns, is around 8500. The Jacobian system in NR is about 17000×17000. The power flow solution associated with this circuit is described as sufficient to exercise most distribution system analysis algorithms and prove the ability to handle large scale problems. The test feeder is provided with balanced $120\,V$ secondary loads on the service transformers. A second load case with random unbalancing is also considered. The convergence for a simple power flow solution with fixed control settings is shown in figure 2 for both NR and the method of alternating search directions. The time unit is taken as the average execution time of a NR iteration. This is done to show how, although
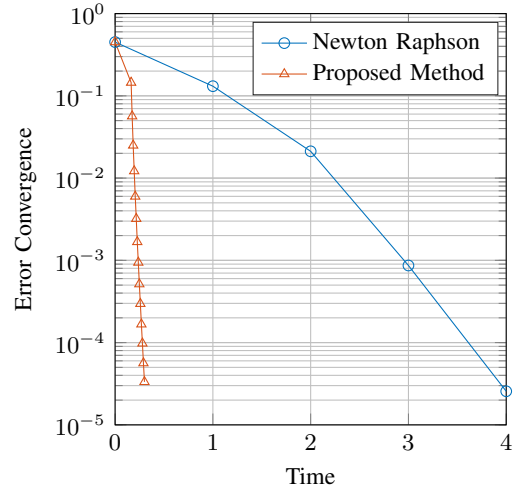

Fig. 2   Comparison between NR and the proposed approach. Convergence is estimated in the infinity norm of the error. The time unit is taken as the average time of a NR iteration.

the number of iterations required is higher, the proposed approaches converges in a fraction of the time required by a single NR iteration.

The circuit includes four voltage regulators that are used to control the feeder voltage at the substation and along the lines. Control actions are taken into account by an external control loop that iterates the power flow solution and changes the tap settings accordingly until the voltages fall within an acceptable range. The factorization of the matrix $(\boldsymbol{Y} - \boldsymbol{\alpha})$ is not performed anew at every control iteration, but the LU factor are updated using the Woodbury formula [34], since the effect of changing the tap settings only changes a few entries in the admittance matrix. The balanced and unbalanced load cases take respectively 6 and 5 control steps.

Different combinations of search directions are presented in table III, along with the number of total iterations and CPU times. The code is run using MATLAB on an Intel i7 (2 GHz) processor. Note that the algorithm $A1$ corresponds to the solver in Open DSS, while $A2$ to IZB. For this particular system and for both the analyzed load cases, $A2$ presents the fastest execution time, because for this case the number of arithmetic operations to perform in the local stage is greatly reduced, whereas the fastest converge rate is obtained with the algorithm $A5$ when the search directions are orthogonal.

V. CONCLUSIONS

In this paper a new class of nonlinear solvers for the power flow problem was introduced. The possibility of tuning the search directions α and β allows achieving a good performance in terms of computational time. A number of iterative schemes can be interpreted as particularizations of the method of alternating search directions including Gauss-Seidel, Newton-Raphson and Implicit Z bus method. In the paper, new search directions are proposed and tested against practical test cases. At each iteration of the proposed method the solution of a linear system is required, but the matrix

Case	$\boldsymbol{\alpha}$	$\boldsymbol{\beta}$	Iterations (Bal)	CPU-time (Bal)	Iterations (UnBal)	CPU-time (UnBal)
$A1$	$diag\left(S^* \oslash |V_n|^2\right)$	∞	67	$0.125s$	69	$0.129s$
$A2$	0	∞	59	$0.102s$	61	0.106
$A3$	$diag\left(S^* \oslash |V_n|^2\right)$	$diag\left(\boldsymbol{Y}\right)$	66	$0.139s$	68	0.144
$A4$	0	$diag\left(\boldsymbol{Y}\right)$	57	$0.119s$	60	$0.126s$
$A5$	$diag\left(\boldsymbol{Y}\right)^{-1}$	$diag\left(\boldsymbol{Y}\right)$	55	$0.117s$	58	$0.124s$

TABLE III   Results of numerical experiments using different combinations of search directions.

is constant through the whole process and factorization is only preformed once in the setup phase. Therefore, for large grids, the method becomes considerably faster than classical NR implementations. The proposed approach can be easily applied even in cases where Fast Decoupled Load Flow is known to converge very poorly, that is, in systems with large resistance to reactance ratios, as it is in most distribution systems. Moreover, unlike FDLF and NR, the convergence to the operative solution does not depend on the choice of the initial guess since the high voltage solution is enforced by construction throughout the whole iterative process. Finally, PV nodes are more easily handled than in the case of similar methods like IZB and reactive power limits can be taken into account within the power flow loop, so that the global number of iteration is kept relatively low.

The search directions provided in this work were found through numerical experimentation. Indeed, the determination of optimal search directions is still an open issue and is the object of ongoing studies. An interesting point, which deserves further investigation, is the possibility of using a variable β^l at each iteration, which would yield non-stationary iterative methods and offer a possible margin for improvement. A further interest of the presented methodology is the possibility of integrating the current algorithm with model order reduction techniques for the calculation of parametric power flow solutions. Ongoing research is currently exploring the combination of this approach with the Proper Generalized Decomposition technique [35]. This is delivering promising results, enabling to produce a full parametric description of the power flow solutions considering the power generation and demand as additional variables of the power flow problem.

APPENDIX

In this appendix the algorithm is explained in detail for the case of diagonal $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ matrices. To simplify the manipulation of the equations, we define B as the vector of the diagonal elements of $\boldsymbol{\beta}$. Once the search directions are fixed, the matrices $\boldsymbol{Y} - \boldsymbol{\alpha}$ and $\boldsymbol{Y} - \boldsymbol{\beta}$ are formed and the LU decomposition of $\boldsymbol{Y} - \boldsymbol{\alpha}$ is performed. Hereafter, the multiplication by the inverse of $\boldsymbol{Y} - \boldsymbol{\alpha}$ indicates the operation of backward and forward substitution for the lower and upper triangular factors of the LU decomposition.

Given the iterate V^l, the algorithm is composed of the following two steps:

- Step one consists in solving of the linear system

$$V^{l+\frac{1}{2}} = \left(\boldsymbol{Y} - \boldsymbol{\alpha}\right)^{-1}\left(S^* \oslash V^{l^*} - \boldsymbol{\alpha}V^l + I_0\right). \quad (27)$$

- Step two involves the solution of n decoupled second order equations

$$B \odot V^{l+1^*} \odot V^{l+1} + V^{l+1^*} \odot \left[\left(\boldsymbol{Y} - \boldsymbol{\beta}\right) V^{l+\frac{1}{2}} - I_0\right] - S^* = 0 \quad (28)$$

Assuming that none of the elements of B is zero, and setting $A = \left[\left(\boldsymbol{Y} - \boldsymbol{\beta}\right) V^{l+\frac{1}{2}} - I_0\right] \oslash B$, $U = V^{l+1} \oslash A$ and $\Sigma = -S^* \oslash \left(A \odot A^* \odot B\right)$, the above equation can be written as

$$U^* \odot U + U^* + \Sigma = 0. \quad (29)$$

Considering the Cartesian form of U and Σ we have

$$\begin{cases} \Re(U)^2 + \Im(U)^2 + \Re(U) + \Re(\Sigma) = 0 \\ -\Im(U) + \Im(\Sigma) = 0 \end{cases}, \quad (30)$$

which yields:

$$U = \frac{-1 \pm \sqrt{1 - 4(\Im(\Sigma)^2 + \Re(\Sigma))}}{2} - i\Im(\Sigma). \quad (31)$$

In the above expression all the operations are intended as component-wise on the vectors U and Σ. In order to reach the high voltage solution, the root corresponding to the sign plus is selected.

Note that the current unknowns I are never explicitly computed during the iterative steps, therefore the number of unknowns is the same as the original problem (3).

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