Numerical representation of the quality measures of triangles and triangular meshes

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SUMMARY

In this note a new procedure to represent the quality measure for triangles is proposed. The triangles are identified by their three angles and are represented in a bounded domain, called angle representation region, according to the area coordinates,which are common and well-known by finite element users. The developed representation can also be used in order to visualize the characteristics of any quality measure. This new procedure is extended to graphically represent triangular meshes in the angle representation region.

KEY WORDS: Finite element meshes, triangular elements, quality measure.

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1. INTRODUCTION

Unstructured triangular meshes have been extensively used in bi-dimensional analysis over the last decades, see references [1, 2] for a detailed review on mesh generation algorithms. It is well known that the error of the finite element analysis depends, among other factors, on the shape of the generated elements. Therefore, all algorithms focus on identifying degenerate triangles (see reference [1] for a classification of triangles according to its shape).

A wide range of measures of quality of triangular elements can be found in the literature [1, 3, 4, 5]. According to [4], a *fair* measure of the quality of a triangle should clearly identify well shaped and distorted elements. Moreover, it should verify the following four properties: *1.*- be able to detect all degenerate elements; *2.*- be independent of the element size, and rigid body translations and rotations (non–dimensionality); *3.*- not yield an arbitrarily large value (boundedness); and *4.*- range from 0 to 1 in order to allow better comparisons between different measures (normalization). The representation of a *fair* measure of the quality of the triangle has to highlight these properties. Moreover, it has to visualize and clearly distinguish the well and bad shaped triangles.

[4] introduces the concept of the Universal Similarity Region (USR). In this representation the quality measure of a triangle and/or the triangle itself can be represented in a bounded region. It is important to note that in the USR representation, each family of triangles (a triangle and all the triangles similar to it) is identified by the length of an edge and two angles.

In this paper a new representation of the quality measures of the triangles is presented. It identifies each family of triangles by its three inner angles. Moreover, it allows to represent all the triangles in a bounded region, called angle representation region (ARR). In this region, contour levels of a quality measure can easily be plotted. Moreover, the developed representation can be efficiently used in order to compare quality measures. And since it is based on the area coordinates, which are well known by the finite element community, insightful conclusions can be gained about contours depicting measures of quality and about quality measures of specific triangles or meshes.

Several quality measures of a triangle have been developed over the past decades [3, 1, 4, 5]. The objective here is not to analyze and compare them. However, some of them will be used to visualize the applicability of the developed representation. In particular, we will use the following measures:

$$\begin{array}{lll} q_{\alpha_{\min}} = \dfrac{3\alpha_{\min}}{\pi} & q_{Ll} = \dfrac{l_{\min}}{l_{\max}} & q_{ALS} = \dfrac{4\sqrt{3}A}{l_1^2 + l_2^2 + l_3^2} \\ q_{Rr} = \dfrac{2r}{R} & q_{Lr} = \dfrac{2\sqrt{3}r}{l_{\max}} & q_{Lh} = \dfrac{2h_{\min}}{\sqrt{3}l_{\max}} \end{array} \tag{1}$$

where $\alpha_{\min}$ is the smallest inner angle; $l_{\min}$ and $l_{\max}$ are the length of the shortest and longest edge respectively; l_1, l_2 and l_3 are the length of the three sides of the triangle; A is the area of the triangular element; r is the inradius of a triangle; R is the circumradius of a triangle; and $h_{\min}$ is the minimum height of the triangle.

2. REPRESENTATION OF A QUALITY MEASURE OF TRIANGLES

2.1. The angle representation region

Since a *fair* measure of quality of triangles should be independent of the element size, rigid body translations and rotations, the new representation is based on the angles of the triangles,

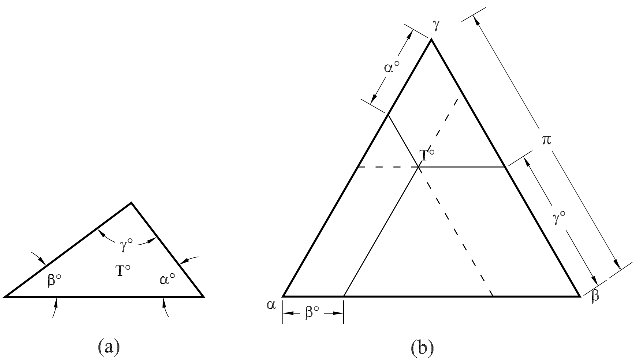

Figure 1. **(a)** A given triangle$T^0\,(\alpha^0, \beta^0, \gamma^0)$ and **(b)** its representation using the area coordinates.

namely: α, β and γ. Note that only two independent angles can be considered since:

$$\alpha + \beta + \gamma = \pi. \tag{2}$$

Therefore, a triangle can be represented by three variables, (α, β, γ), and a linear constraint, defined by equation (2). Or, equivalently, using area coordinates [6]. Each triangle thus can be represented by a unique point within the equilateral triangle commonly used to plot area coordinates. Figure 1.a shows a given triangle $T^0\,(\alpha^0, \beta^0, \gamma^0)$, and figure 1.b shows its representation (the point T^0) using area coordinates.

The basic triangles can be easily identified in the angle representation region (ARR). The equilateral triangle corresponds to the point $(\alpha, \beta, \gamma) = (\pi/3, \pi/3, \pi/3)$ (see figure 2.a). The isosceles triangles, α being the different angle, are placed on the bisecting line of vertex α (see segment $\overline{\alpha\alpha\prime}$ in figure 2.b). The same property applies for isosceles triangles, where the different angle is β or γ. Note that these three bisecting lines divide the ARR in six equal (or equivalent) regions. Degenerate triangles are placed on the boundary of the ARR. For instance, triangles with $\alpha = \pi$ are represented at vertex α. Moreover, if α tends to zero, the representation of the triangle tends to the edge $\overline{\beta\gamma}$.

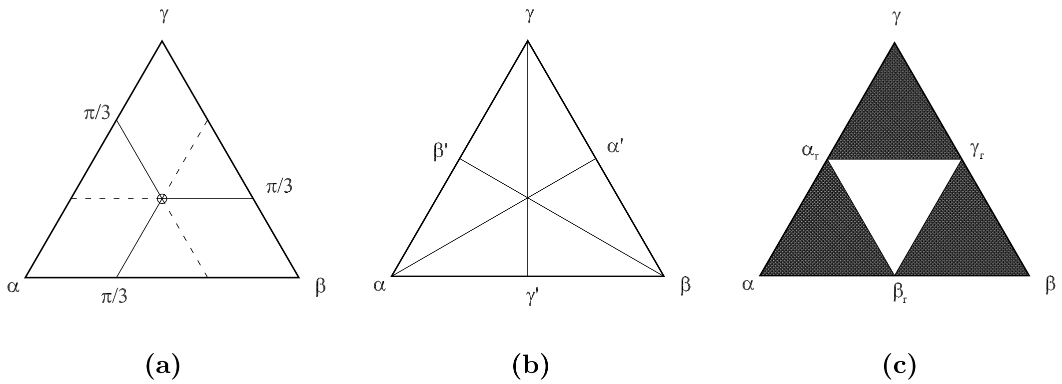

Figure 2. **(a)** Representation of an equilateral triangle, **(b)** Representation of isosceles triangles, **(c)** Partition of the ARR in two disjoint regions corresponding to triangles with an obtuse inner angle (shadowed region) and triangles with only acute inner angles (white region).

It is straightforward to identify the regions corresponding to acute and obtuse angles in the ARR. Let $\alpha_r = \pi/2$, $\beta_r = \pi/2$, and $\gamma_r = \pi/2$ denote the right angles for α, β, and γ respectively (see figure 2.c). All triangles with the right angle equal to $\alpha = \alpha_r$ are placed on the segment $\overline{\alpha_r\beta_r}$ (see figure 2.c). Note that similar segments can be defined for right angles β_r and γ_r. Therefore, the ARR can be subdivided in two disjoint regions. The first region corresponds to triangles with an obtuse inner angle (shadowed region in figure 2.c). The second region corresponds to triangles with only acute inner angles (white region in figure 2.c).

2.2. Representation of a measure of quality

It is extremely simple to represent a measure of the quality of a triangle in the ARR. Given a quality measure, for each triangle (for each point on the ARR) a scalar value can be computed. Thus, it is possible to draw contour levels of this measure of quality. The contour level of any *fair* measure of the quality of a triangle has to meet the following three conditions in terms of its representation in the ARR:

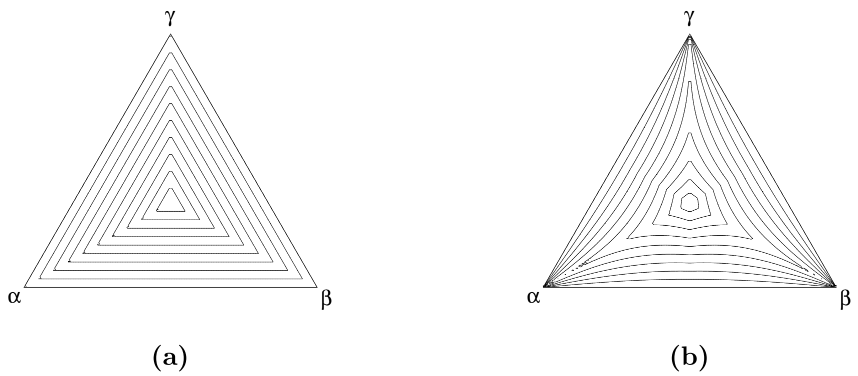

Figure 3. Representation in the ARR of two quality measures. Contour levels range form 0 to 1 with an increment of 0.1 **(a)** $q_{\alpha_{\min}}$ **(b)** q_{Ll}

- The contour level corresponding to $q = 1$ (maximum quality) is a point placed at the center of the ARR, $(\alpha, \beta, \gamma) = (\pi/3, \pi/3, \pi/3)$. This corresponds to a equilateral triangle.
- The contour level corresponding to $q = 0$ (triangles that degenerate to a segment) is the boundary of the ARR.
- If q_1 and q_2 are two contour levels with $0 \leq q_1 \leq q_2 \leq 1$, then the contour level corresponding to q_2 must be included in the inner region bounded by q_1. This condition implies that any fair measure will decrease monotonically from the center (the optimal triangle) to the boundary (degenerate triangles) of the ARR.

Figure 3.a shows the contour levels corresponding to the measure of quality $q_{\alpha_{\min}}$, see (1). In this case, all previous conditions are fulfilled. The contour levels corresponding to the measure of quality q_{Ll} are plotted in Figure 3.b. Note that now, the previous conditions are not met. Degenerate triangles with an inner angle equal to π have a quality measure different from zero. That is, contour levels converge to cusp at the corners of the ARR.

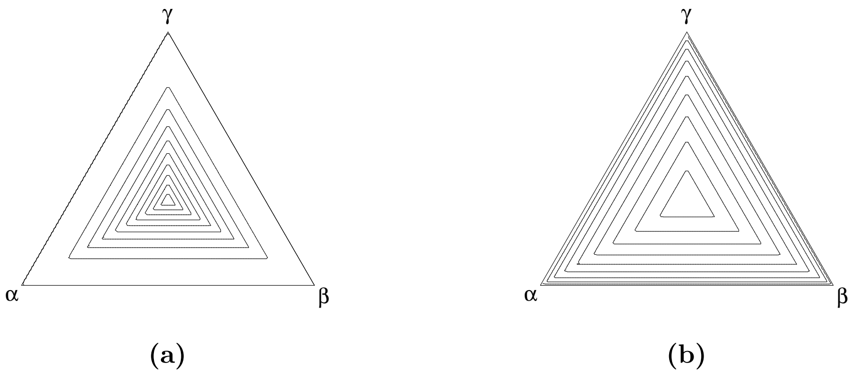

Figure 4. Representation in the ARR of a quality measure. Contour levels range form 0 to 1 with an increment of 0.1 **(a)** $\tilde{q} = q_{\alpha_{\min}}^2$, **(b)** $\tilde{q} = \sqrt{q_{\alpha_{\min}}}$.

It is important to note that, on one hand, every quality measure defines the shape of its contour levels in the ARR. For a given triangle, a quality measure may be prone to penalize one kind of distortion more than other. The weight of each kind of distortion is an intrinsic property of each quality measure, and it is reflected in the shape of the contour levels. On the other hand, the separation between contour levels in the ARR can be easily modified and it it not an intrinsic property of the quality measure. Consider a quality measure q. Let $F(x)$ be a monotone function that maps $[0, 1]$ into $[0, 1]$. Then $\tilde{q} = F(q)$ is another quality measure whose contour levels have the same shape as those of q. However, the separation between contour levels is different.

For instance, compare Figure 3.a where $q_{\alpha_{\min}}$ is used with in figure 4.a and 4.b, where the quality measures are $\tilde{q} = q_{\alpha_{\min}}^2$ and $\tilde{q} = \sqrt{q_{\alpha_{\min}}}$ respectively. Notice that in figure 4.a the contour levels are concentrated near the center of the ARR and that in figure 4.b the contour levels are moved towards the boundary of the ARR. This may be of the major importance when a measure is used to optimize the quality of the mesh.

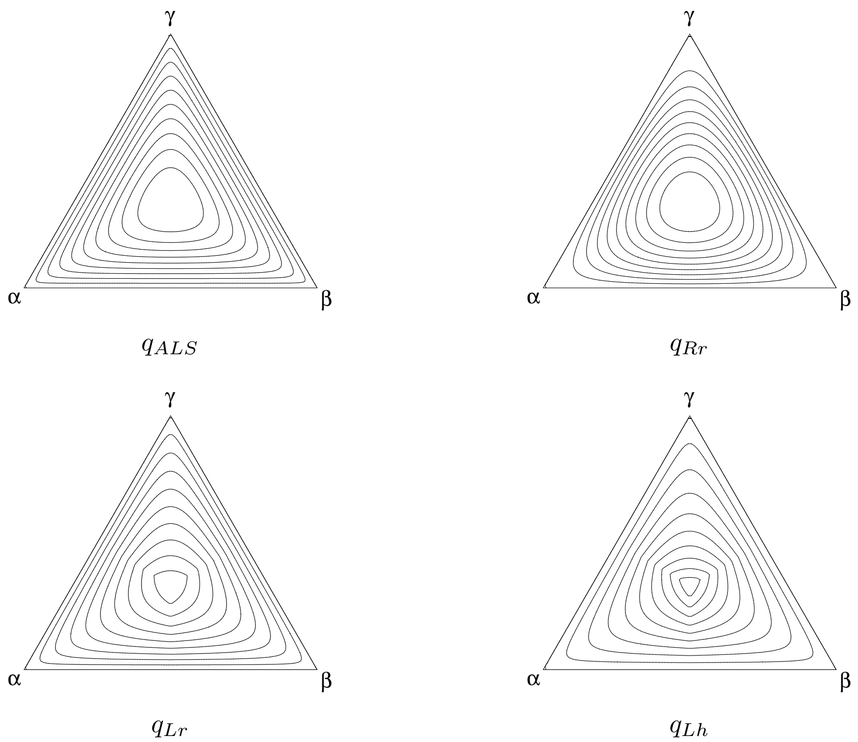

Figure 5. Representation in the ARR of a quality measure. Contour levels range form 0 to 1 with an increment of 0.1

Figure 5 shows the contour levels of the measures q_{ALS}, q_{Rr}, q_{Lr}, and q_{Lh} in the ARR. On one hand, the contour levels corresponding to q_{ALS}, and q_{Rr} have a similar shape. Both quality measures are almost constant near the center of the ARR (equilateral triangles), and the gradient of the contour levels increase significantly near its boundary (degenerate triangles). This behavior is more pronounced in the case of the q_{ALS} measure. On the other hand, q_{Lr} and q_{Lh} have similar contour levels with a shape similar to an inverted triangle near the center of the ARR. Both measures present an almost linear variation of the contour level separation. Note, that q_{Lr} and q_{Lh} show a discontinuity in the slope of the contour levels for isosceles triangles. This property also appear in other quality measures that involves minimum or maximum values. Finally, it is important to note that the contour levels corresponding to the q_{Rr} measure are more separated near the vertices of the ARR. This reflects that this measure, compared with the others, is less sensible for triangles with an inner angle close to π.

3. REPRESENTATION OF A QUALITY MEASURE OF MESHES

Given a triangular mesh, it can also be represented in the ARR. Each element (triangle) will be represented as a point in the ARR. In order to generate a coherent representation, the inner angles of each triangle are sorted in a decreasing order, for instance: $\alpha \geq \beta \geq \gamma$. Note that, if triangles are not sorted, they will be placed around the barycenter of the ARR, and it will be difficult to identify any dominant trend. Once an order has been fixed, the mesh will be represented in one sixth of the angle representation region, see figures 6.b, 7.b, 8.b and 9.b (in this figures point $\tilde{\alpha}_r$ is the intersection of the bisecting line $\overline{\alpha\alpha\prime}$ of figure 2.b and segment $\overline{\alpha_r\beta_r}$ of figure 2.c). Recall that the bisecting lines of the angles defined by vertices α, β and γ, see figure 2.b, split the ARR in six equivalent subregions.

The main objective here is to illustrate that the developed representation can be a useful and helpful tool if it is incorporated in a mesh generation software. To this end we will present four triangulations of a square domain obtained by different mesh generation algorithms. It is important to note that the developed representation clearly distinguishes them and contains, for instance, the information represented in histograms.

In order to gather all the information contained in the new representation of the mesh quality it is important to keep in mind the following six properties, which are well known by the finite element community comfortable with the area coordinates: *1.-* If the value of the biggest angle is π, then the triangle will be placed at vertex α (see figure 1.b). *2.-* An equilateral triangle will be represented on vertex O (see figure 2.a). *3.-* The line $\overline{\tilde{\alpha}_r\beta_r}$ separates triangles with an obtuse inner angle and triangles with only acute inner angles (see figure 2.c). *4.-* Isosceles triangles, α (the biggest angle) being the different angle, will be plotted on segment $\overline{\alpha O}$ (see figure 2.b). *5.-* Isosceles triangles, γ (the smallest angle) being the different angle, will be plotted on segment $\overline{\beta_r O}$ (see figure 2.b). *6.-* Contour levels of any fair quality measure of triangles vary from the boundary of the ARR ($q = 0$ on the segment $\overline{\alpha\beta_r}$), to vertex O where they reach their maximum value ($q = 1$).

Figure 6.a shows the first triangulation of a square domain with high nodal density prescribed at two opposed vertices. The mesh is obtained using the advancing front method and it is composed by 676 nodes and 1242 elements. In figure 6.b the previous mesh is represented in one sixth of the ARR. In order to visualize the quality of the triangles, the contour levels corresponding to the q_{ALS} measure are also plotted. Note that most of the triangles have three acute angles and are placed near the equilateral triangle. In fact, only few triangles appears with the biggest angle greater than $\pi/2$. Moreover, since most of them are plotted near the segment $\overline{\beta_r O}$, we can conclude that most of them are almost isosceles triangles being γ (the smallest angle) the different angle. Therefore, all the elements of the mesh are well shaped triangles, $q_{ALS} \in (0.9, 1.0)$. Moreover, there are no triangles with $q_{ALS} \leq 0.5$. It is important to note that this information is concentrated in only one figure and that it is in concordance with the histograms that shows the distribution of the quality of the triangles and the distribution of the biggest angle (figures 6.c and 6.d respectively).

Figure 7.a shows the triangulation of the same square domain when it is meshed using a non

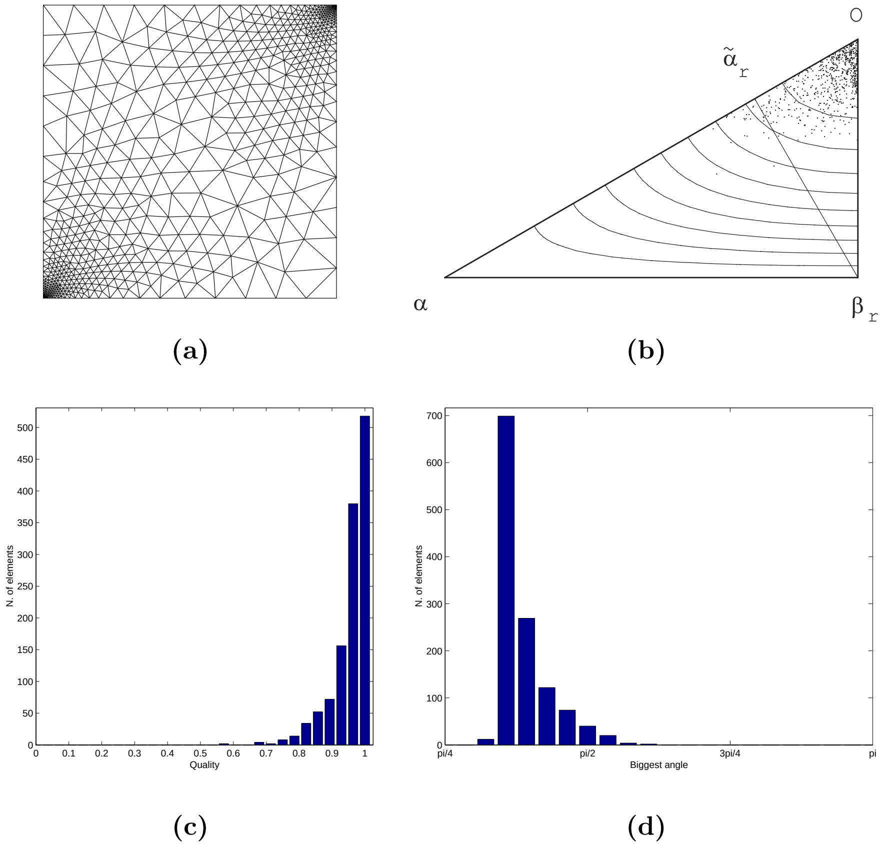

Figure 6. Representation of a meshes in the ARR: **(a)** triangular mesh, **(b)** contour levels of q_{ALS}, **(c)** distribution of the quality of the elements, **(d)** distribution of the biggest angle of the elements.

isotropic element size. The mesh is composed by 754 nodes and 1368 elements. Figure 7.b plots the triangles on the ARR, also superposed are contour levels corresponding to q_{ALS}. Note that this representation points out that this mesh lacks of equilateral triangles. In fact, there are far less triangles with all three acute angles than in the previous case. A sparse distribution of the triangles in the ARR is obtained. It is important to note that $q_{ALS} \in (0.4, 0.8)$. In particular, there is a triangle, which is almost isosceles, with an obtuse angle around $\alpha \approx 8\pi/9$ and $q_{ALS} \approx 0.05$. As in the previous example, these characteristics of the generated mesh are

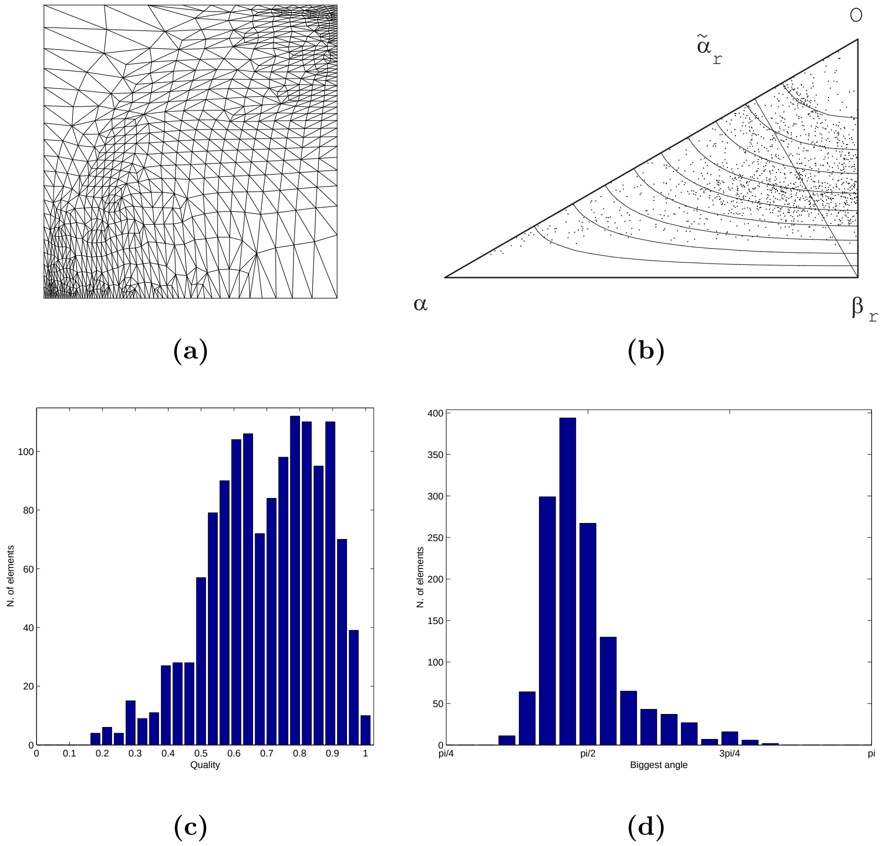

Figure 7. Representation of a meshes in the ARR: **(a)** triangular mesh, **(b)** contour levels of q_{ALS}, **(c)** distribution of the quality of the elements, **(d)** distribution of the biggest angle of the elements.

well represented in the ARR. The histograms of the distribution of the quality of the triangle and its biggest angle, figures 7.c and 7.d, corroborate this result.

Figure 8.a shows the third triangulation. The mesh is composed by 728 nodes and 1318 elements and it is obtained in two steps. First, quadrilaterals elements are generated [7]. Second, each quadrilateral is subdivided in two triangles joining the first and third nodes of the quadrilateral element. Figure 8.b shows the ARR representation and the contour levels of q_{ALS}. In this case most of the triangles are placed near the segment $\overline{\alpha O}$. Therefore, most of the

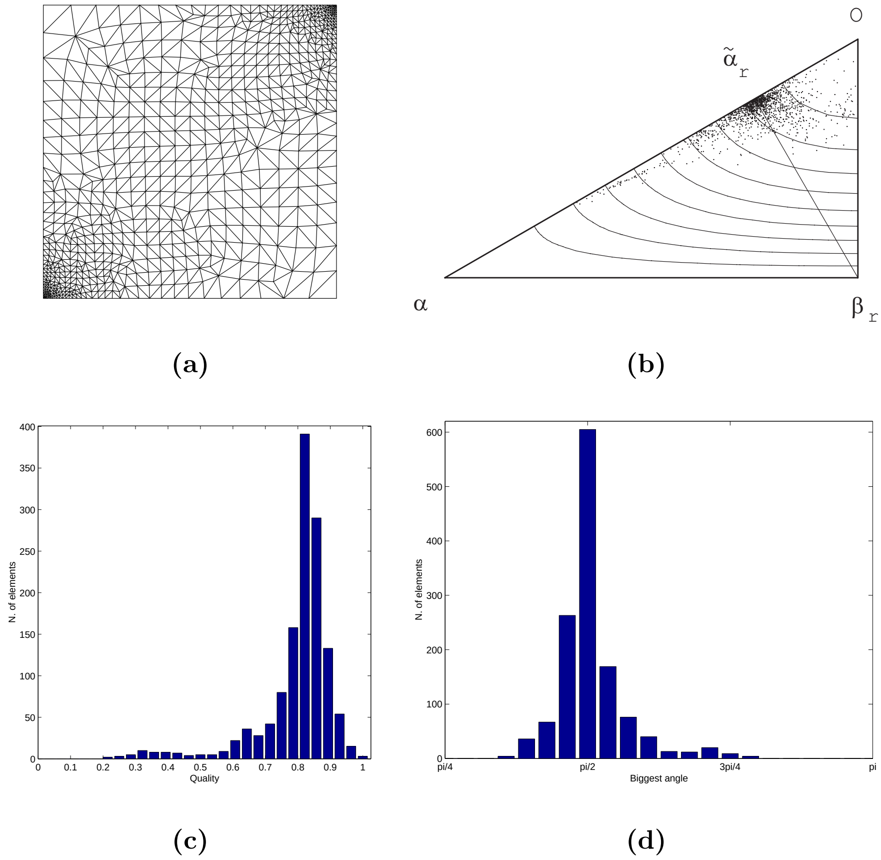

Figure 8. Representation of a meshes in the ARR: **(a)** triangular mesh, **(b)** contour levels of q_{ALS}, **(c)** distribution of the quality of the elements, **(d)** distribution of the biggest angle of the elements

triangles are isosceles with α being the different angle and largest angle. Moreover, the elements are concentrated in a cloud close to $\tilde{\alpha}_r$. Therefore, α, the different angle, is approximately $\pi/2$. Most of the triangles lie in the gap $q_{ALS} \in (0.8, 0.9)$, and the lowest measure, $q_{ALS} \approx 0.2$, is associated to an almost isosceles triangle with $\alpha \approx 5\pi/6$. Again, this information is consistent with the histograms presented in figures 8.c and 8.d.

Finally, figure 9.a shows the fourth triangulation. This mesh is composed by the same

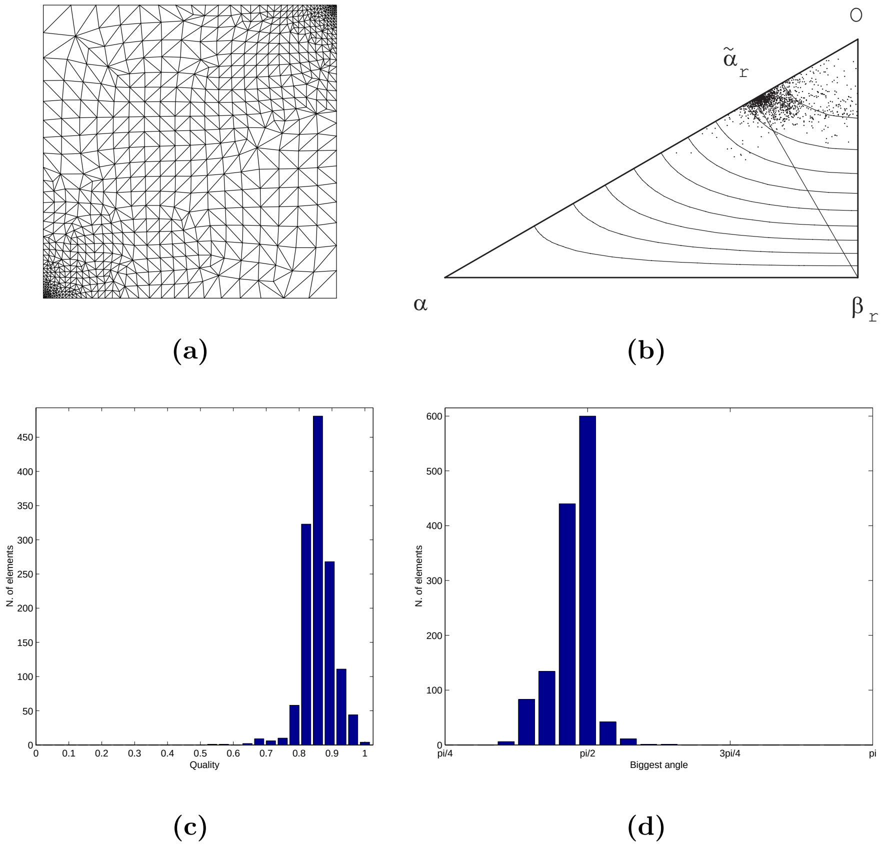

Figure 9. Representation of a meshes in the ARR: **(a)** triangular mesh, **(b)** contour levels of q_{ALS}, **(c)** distribution of the quality of the elements, **(d)** distribution of the biggest angle of the elements.

728 nodes of the previous example. Connectivity is defined using a standard Delaunay triangulation. Figure 9.b shows the ARR representation and q_{ALS}. The cloud of triangles is now more concentrated and closer to vertex O because Delaunay favors equilateral triangles. Therefore, the quality of the triangles also improves. This behavior can be also observed in the histograms presented in figures 9.c and 9.d.

4. CONCLUSIONS

This paper proposed a new procedure to represent the quality measure for triangles. It identifies each family of triangles by the area coordinates. They are represented in a bounded domain called the angle representation region. Basic regular triangles, such as equilateral and isosceles triangles, can be easily identified in the angle representation region. Moreover, since degenerate triangles lies on its boundary, the new representation clearly distinguish them. Contour levels corresponding to different quality measures can be plotted easily using this representation. This new procedure can also be used in order to graphically represent triangular meshes in the angle representation region. Insightful conclusions can be gained about contours depicting quality measures and about quality measures of specific triangles or meshes, in particular for the finite element community familiar with area coordinates.

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