The 9th European Congress on Computational Methods in Applied Sciences and Engineering
ECCOMAS Congress 2024
3–7 June 2024, Lisboa, Portugal

CALIBRATION OF HESTON STOCHASTIC LOCAL VOLATILITY MODEL BY NUMERICAL SOLUTION OF NONLINEAR FOKKER-PLANCK EQUATION

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Key words: Non-linear time-dependent PDE, Adaptive time step, Adaptive mesh, B-Spline discretization

Summary. The Fokker-Planck partial differential equation is used to compute the probability density function of the Heston stochastic local volatility model. The solution of the Fokker-Planck equation is required for the calibration of the leverage function, which plays an important role in the Heston stochastic local volatility model. The current study describes a numerical method for solving the nonlinear Fokker-Planck partial differential equation. The solution is demonstrated to converge to the one generated from the implied volatility surface by comparing call option prices.

1 INTRODUCTION

The Fokker-Planck partial differential equation (PDE) defines the time evolution of the probability density function of a stochastic process. When the stochastic process is distribution-dependent, the resulting Fokker-Planck PDE is nonlinear. The current work presents a numerical solution to a two-dimensional nonlinear Fokker-Planck PDE using a variational approach. B-Splines with non-uniform knot vectors are used for space discretization.

Implicit Runge-Kutta methods are used to solve the obtained nonlinear system of ordinary differential equations in the time domain. The initial condition, which is the Dirac delta function, requires a fine mesh of knots around the non-zero domain of the Dirac delta function. Furthermore, the time discretization method requires very small time steps to successfully handle the evolution of the initial condition. Since the Fokker-Planck equation has a diffusion term, the probability density function evolves with time, and the fine mesh of knots is not required as it is at the initial time steps. The numerical solution employs adaptive mesh generation and adaptive time steps for both the space and the subsequent time discretization of the PDE. The numerical experiments are presented using the Heston stochastic local volatility model, which is a widely used stochastic process for pricing exotic options.

2 HESTON STOCHASTIC LOCAL VOLATILITY MODEL

The following system of stochastic differential equations expresses the dynamic of the Heston stochastic local volatility model:

$$\begin{aligned} dS_t &= \mu S_t dt + L(S_t, t)\sqrt{V_t} S_t dW_t^S \\ dV_t &= \kappa(\theta - V_t)dt + \eta\sqrt{V_t} dW_t^V \end{aligned} \tag{1}$$

where S_t represents the price of the underlying asset, V_t represents the variance of the asset and μ is the rate of return of the asset. θ is long run variance, κ is mean-reversion rate, η is volatility of volatility, W_t^S and W_t^v are Wienner processes with correlation ρ. These parameters, together with V_0, which denotes the initial variance, are the Heston parameters.

$L(s,t)$ is the leverage function, which is important in the model above because its successful calibration allows one to match the volatility surface from the market data.

3 CALIBRATION OF THE LEVERAGE FUNCTION

The leverage function is related to the local volatility function by the mimicking theorem:

$$L(s,t) = \frac{\sigma_{LV}(s,t)}{\sqrt{E[v|s]}} = \sigma_{LV}(s,t)\sqrt{\frac{\int_0^\infty p(s,v,t)dv}{\int_0^\infty v p(s,v,t)dv}} \tag{2}$$

where $p(s,v,t)$ denotes the density function of the probability distribution of the stochastic process defined in eq. 1 and $\sigma_{LV}(s,t)$ represents the local volatility function. The probability density function can be obtained by solving the Fokker-Planck equation.

3.1 Fokker-Planck equation

In stochastic processes, the Fokker-Planck equation is a partial differential equation that characterizes the temporal development of the probability density function. The density function $p(s,v,t)$ of the stochastic process given in eq. 1 is defined by the following equation [1]:

$$\begin{aligned} \frac{\partial p}{\partial t} + \nabla \cdot \mathbf{J} &= 0 \\ p(s,v,0) &= \delta_0(s,v) \end{aligned} \tag{3}$$

where $\mathbf{J}(s,v,t)$ represents the probability current, defined to be:

$$\mathbf{J} = \mathbf{b}p - \frac{1}{2}\nabla \cdot (\mathbf{\Sigma} p) \tag{4}$$

$\delta_0(s,v)$ represents the initial condition of the probability density function, $\mathbf{b}$ represents the drift vector, and $\mathbf{\Sigma}$ represents the diffusion matrix.

The diffusion matrix of the systems of stochastic differential equations, eq. 1, is defined as:

$$\mathbf{\Sigma} = \begin{bmatrix} L(s,t)\sqrt{v}s & 0 \\ 0 & \eta\sqrt{v} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{bmatrix}^\top \begin{bmatrix} L(s,t)\sqrt{v}s & 0 \\ 0 & \eta\sqrt{v} \end{bmatrix}^\top = \begin{bmatrix} L(s,t)^2 v s^2 & L(s,t)\rho\eta v s \\ L(s,t)\rho\eta v s & \eta^2 v \end{bmatrix} \tag{5}$$

and the drift vector is defined by:

$$\mathbf{b} = \begin{pmatrix} \mu s \\ \kappa(\theta - v) \end{pmatrix} \tag{6}$$

The Fokker-Planck equation is formulated with reflecting boundary conditions [2]. The criteria indicate that the particle, which in the present application denotes an element from the price-variance space, is confined within the domain Ω and cannot leave it. This is expressed as a zero net flow of probability across $\partial\Omega$:

$$\mathbf{J}(s, v, t) \cdot \boldsymbol{n} = 0, (s, v) \in \partial\Omega \tag{7}$$

where $\boldsymbol{n}$ is the normal vector to $\partial\Omega$. In the variational formulation of the problem, the reflecting boundary conditions are called natural boundary conditions.

The probability current $\mathbf{J}$, defined on the system of stochastic differential equations, eq. 1, has the following expression:

$$\begin{aligned} \mathbf{J}_1 &= \mu s p(s, v, t) - \frac{1}{2}\frac{\partial}{\partial s}\left(L(s, t)^2 s^2 v p(s, v, t)\right) - \frac{1}{2}\rho\eta\frac{\partial}{\partial v}\left(L(s, t) s v p(s, v, t)\right) \\ \mathbf{J}_2 &= \kappa(\theta - v) p(s, v, t) - \frac{1}{2}\rho\eta\frac{\partial}{\partial s}\left(L(s, t) s v p(s, v, t)\right) - \frac{1}{2}\eta^2\frac{\partial}{\partial v}\left(v p(s, v, t)\right) \end{aligned} \tag{8}$$

3.2 Space discretization

Finite difference methods and variational methods [3] are among the most common techniques used to solve partial differential equations. However, although more intricate from a development standpoint, variational approaches offer certain benefits, which are detailed below. By transforming the partial differential equation into its weak form, the variational method seeks the solution as a linear combination of independent shape functions. Unlike finite difference techniques, which only yield solutions at discrete nodes, this approach enables differentiation and integration of the solution. Implementing boundary conditions that involve derivatives on finite difference schemes is challenging, whereas variational approaches can readily handle all kinds of boundary conditions. The present study employs variational techniques to achieve spatial discretization of the Fokker-Planck equation.

The solution of the partial differential equation is approximated by the following expression:

$$p(s, v, t) \approx \sum_{i=1}^{k} \varphi_i(s, v) q_i(t) \tag{9}$$

where $\varphi_i(s, v)$ are shape (or trial) functions and $q_i(t)$ are functions of time, known as generalized coordinates. The shape functions define a finite dimensional space $V_h = \text{span}\{\varphi_i(s, v)\}$, where the solution is sought.

The partial differential equation, eq. 1, has to be zero at each point of the domain Ω. Then, it follows that:

$$\int_\Omega w(s, v)\left(\frac{\partial p(s, v, t)}{\partial(t)} + \nabla \cdot \mathbf{J}(s, v, t)\right) d\Omega = 0 \tag{10}$$

for any arbitrary function $w(x) \in V_h$. Integration by part leads to the following weak formulation of the problem:

$$\int_\Omega w(s,v)\frac{\partial p(s,v,t)}{\partial(t)}d\Omega - \int_\Omega \nabla w(s,v)\cdot\mathbf{J}(s,v,t)d\Omega + \int_{\partial\Omega} w(x)\mathbf{J}(s,v,t)\cdot\boldsymbol{n}d\Omega = 0 \tag{11}$$

Eq. 11 represents the variational form of eq. 1. The last term of eq. 11 might be omitted since it is satisfied by the natural boundary conditions 7.

A system of first-order ordinary differential equations is derived by expressing the arbitrary functions $w_i(s,v) \in V_h$ using the shape functions $\varphi_i(s,v)$ substituting equation 9 into equation 11, and integrating the resulting equations:

$$\begin{aligned} \mathbf{M}\dot{\mathbf{q}}(t) + \mathbf{K}\left(\mathbf{q}(t)\right)\mathbf{q}(t) &= \mathbf{0} \\ \mathbf{q}(0) &= \mathbf{q}_0 \end{aligned} \tag{12}$$

where vector $\mathbf{q}_0$ represents the initial condition determined from $\delta_0(s,v)$ and the basis of shape functions $\{\varphi_i(s,v)\}$. The matrix $\mathbf{M}$ is often called a mass matrix, and the matrix $\mathbf{K}$ is often called a stiffness matrix. The stiffness matrix depends on the vector of generalized coordinates $\mathbf{q}(t)$ due to the leverage function, which depends on the density $p(s,v,t)$.

A crucial aspect of the suggested variational formulation concerning the precision of the numerical solution is the careful selection of shape functions. Within the present implementation of pricing derivatives, the initial condition denotes the present spot price of the underlying asset, meaning that all probability mass is concentrated at a single location. Hence, $\delta_0(s,v)$ represents the Dirac delta function. B-Splines with small support are a viable category of functions capable of approximating a Dirac delta function. Hence, numerical discretization is performed using B-Splines.

The dimension of the discretized problem is determined by the quantity of shape functions employed for the discretization of the density function. Effective approximation of the Dirac delta function requires the use of B-Splines with a fine mesh. The optimal fine mesh should be in proximity to the spot price rather than across the entire region. Therefore, it is appropriate to employ a non-uniform mesh for the B-Splines. Figure 1 illustrates a two-dimensional B-Spline and an example of non-uniform knots in two-dimensional space.

3.3 Numerical solution of ordinary differential equations

Numerical solution of the system of ordinary differential equations defined by equation 12 is obtained using the Crank-Nicolson method. The nonlinearity of the system of ordinary differential equations leads to a corresponding nonlinearity in the resulting algebraic system of equations. The solution to the nonlinear system is derived by a series of internal loops, wherein the leverage function is updated and the resulting stiffness matrix is also updated. If the nonlinear solution fail to reach convergence within a small number of internal loops, the time step is decreased.

The time discretization method needs a small time step because the initial condition is the Dirac delta function. After a certain point, the time step doesn't have to be as small. An additive time step is built into the time discretization method to reduce the computational time.

The initial condition also requires a fine mesh of knots around the spot price and the initial variance. As the method goes on, this fine mesh is no longer needed because of the diffusion process. Because of this, an adaptive mesh is also used in the numerical answer.

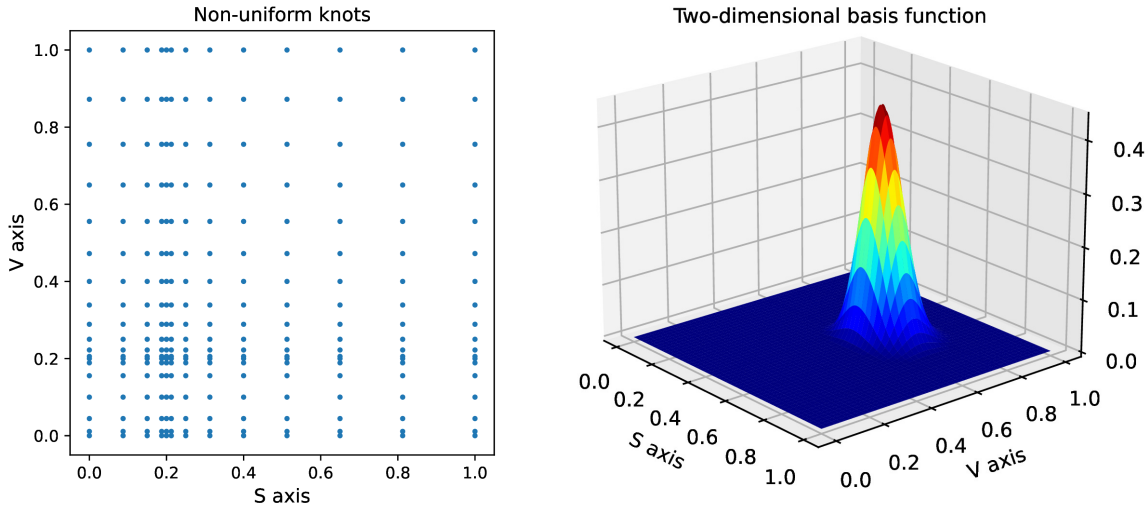

Figure 1: Non-uniform knots in two-dimensional space and an example of a two-dimensional B-Spline.

The probability density function $p(s, v, t)$ is derived continuously at each time step. Next, the leverage function is calculated using equation 2. The local volatility, a component of the leverage function, is derived from the implied volatility surface. The leverage function as calculated is kept in a two-dimensional array, which may be utilized, for instance, in Monte Carlo simulations.

4 RESULTS

An obstacle in solving the Fokker-Planck partial differential equation numerically is the optimal choice of the standard deviation to approximate the initial condition that represents the Dirac delta function. Several two-dimensional normal distributions with varying standard deviations are employed as initial condition, and the related solutions of the partial differential equation are compared.

Figure 2 presents a comparison of the marginal distributions of the asset's price computed by employing various standard deviations to approximate the original situation. The standard deviation values of the initial condition are provided from the local coordinate system. Achieving convergence of the results requires approximating the initial condition using two-dimensional normal distributions with a very low standard deviation.

The prices of the respective call options are shown in Figure 3. The reference solution presents results from the Black-Scholes formula that utilize the same implied volatility surface as the one used in the computation of the local volatility surface $\sigma_{LV}(s, t)$ provided in equation 2. The call option prices derived from the numerical solution of the probability density function show a very good agreement with the call option prices obtained from the volatility surface.

Figure 4 provides a comparison between the number of knots allocated to define the B-Spline basis of shape functions. The knots in the figure labels indicate the number of knots in one dimension. It is evident that using inadequate B-Splines results in an incorrect estimation of the density function, as well as the resultant solution exhibits unsatisfactory oscillations.

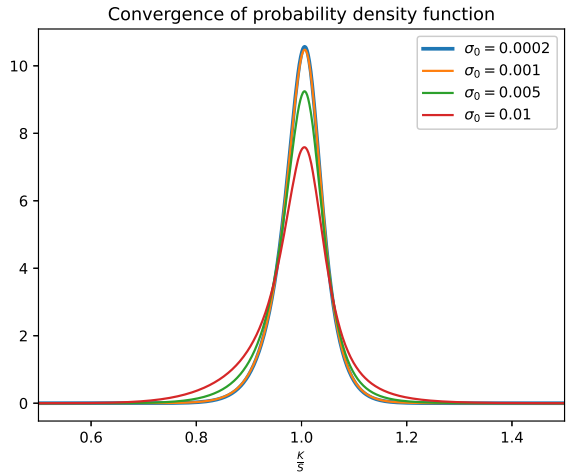

Figure 2: Probability density function due to initial conditions with different standard deviations, $T = 0.5$.

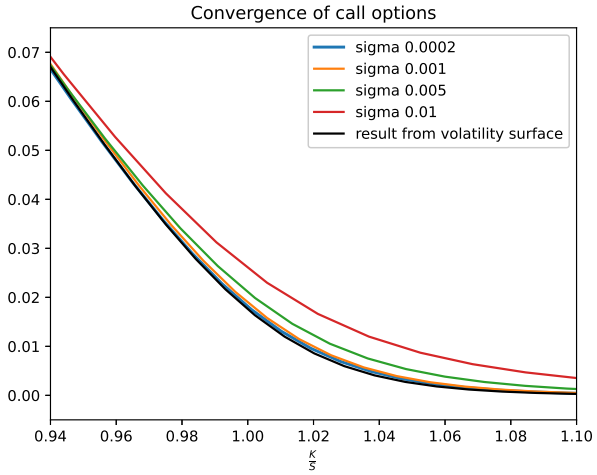

Figure 3: Comparison of call option prices computed from the marginal distribution due to different initial conditions.

5 CONCLUSIONS

This paper presents a numerical approach for solving the nonlinear Fokker-Planck partial differential equations. The Fokker-Planck partial differential equation (PDE) is employed to characterize the probability density function of the Heston stochastic local volatility model.

The Fokker-Planck partial differential equation is discretized using a variational approach. Non-uniform B-Splines are used as shape functions for the space discretization. The numerical solution uses an adaptable mesh and an adaptive time step to reduce the computational time while also accurately approximating the time evolution of the initial condition.

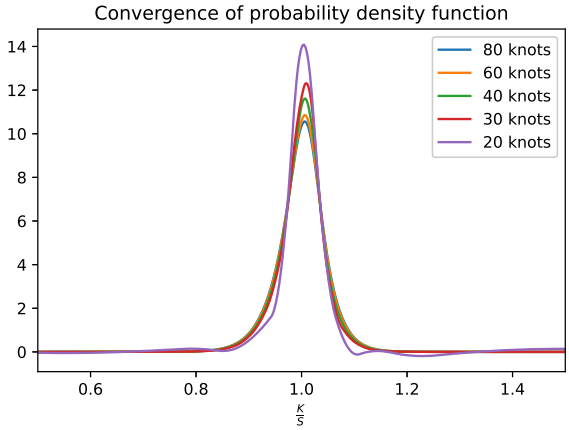

Figure 4: Probability density function with different number of knots, $T = 0.5$.

It is shown that the solution converges to the one derived from the implied volatility surface by comparing the prices of call options.

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